

# COUNTING IMAGINARY QUADRATIC POINTS VIA UNIVERSAL TORSORS, II

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ABSTRACT. We prove Manin’s conjecture for four singular quartic del Pezzo surfaces over imaginary quadratic number fields, using the universal torsor method.

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## 1. INTRODUCTION

Let  $K$  be a number field,  $S$  a del Pezzo surface defined over  $K$  with only **ADE**-singularities,  $U$  the open subset obtained by removing the lines from  $S$ , and  $H$  a height function on  $S$  coming from an anticanonical embedding. If  $S(K)$  is Zariski dense in  $S$  then generalizations (e.g. [BT98b]) of Manin’s conjecture [FMT89, BM90] predict an asymptotic formula, as  $B \rightarrow \infty$ , for the quantity

$$N_{U,H}(B) := |\{\mathbf{x} \in U(K) \mid H(\mathbf{x}) \leq B\}|,$$

namely

$$N_{U,H}(B) = c_{S,H} B(\log B)^{\rho-1} (1 + o(1)),$$

where  $\rho$  is the rank of the Picard group of a minimal desingularization of  $S$  and  $c_{S,H}$  is a positive real number.

Much progress was made in recent years in proving Manin’s conjecture for specific del Pezzo surfaces over  $\mathbb{Q}$  via the *universal torsor method*. In [DF13], the authors extended this method to imaginary quadratic fields in case of a quartic del Pezzo surface of type  $\mathbf{A}_3$  with five lines.

In the present article, we continue this investigation by proving Manin’s conjecture over imaginary quadratic fields for quartic del Pezzo surfaces of types  $\mathbf{A}_3 + \mathbf{A}_1$ ,  $\mathbf{A}_4$ ,  $\mathbf{D}_4$ , and  $\mathbf{D}_5$ .

For more information about Manin’s conjecture and the universal torsor method, we refer to the introductory section of [DF13] and the references mentioned there.

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**1.1. Results.** Let  $K$  be an imaginary quadratic field. We define the anticanonically embedded quartic del Pezzo surfaces  $S_i \subset \mathbb{P}_K^4$  over  $K$  by the following equations:

$$S_0 : \quad x_0x_1 - x_2x_3 = x_0x_3 + x_1x_3 + x_2x_4 = 0 \quad \text{of type } \mathbf{A}_3 \text{ (5 lines)} \quad (1.1)$$

$$S_1 : \quad x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0 \quad \text{of type } \mathbf{A}_3 + \mathbf{A}_1, \quad (1.2)$$

$$S_2 : \quad x_0x_1 - x_2x_3 = x_0x_4 + x_1x_2 + x_3^2 = 0 \quad \text{of type } \mathbf{A}_4, \quad (1.3)$$

$$S_3 : \quad x_0x_3 - x_1x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0 \quad \text{of type } \mathbf{D}_4, \quad (1.4)$$

$$S_4 : \quad x_0x_1 - x_2^2 = x_3^2 + x_0x_4 + x_1x_2 = 0 \quad \text{of type } \mathbf{D}_5. \quad (1.5)$$

All of them are split over  $K$ , hence rational over  $K$ , and therefore, their rational points over  $K$  are Zariski dense. The Weil height on  $\mathbb{P}_K^4(K)$  is defined by

$$H(x_0 : \dots : x_4) := \frac{\max\{\|x_0\|_\infty, \dots, \|x_4\|_\infty\}}{\mathfrak{N}(x_0\mathcal{O}_K + \dots + x_4\mathcal{O}_K)}, \quad (1.6)$$

where  $\mathcal{O}_K$  is the ring of integers in  $K$ ,  $\|\cdot\|_\infty := |\cdot|^2$  is the square of the usual complex absolute, and  $\mathfrak{N}\mathfrak{a}$  is the absolute norm of a fractional ideal  $\mathfrak{a}$ .

For  $S_0$ , Manin's conjecture was proved over  $\mathbb{Q}$  and imaginary quadratic fields in [DF13]. For  $S_1$ ,  $S_2$ ,  $S_3$ ,  $S_4$ , Manin's conjecture was proved over  $\mathbb{Q}$  in [Der09], [BD09], [DT07], [BB07], respectively. In this article, we prove Manin's conjecture for  $S_1, \dots, S_4$  over imaginary quadratic fields:

**Theorem 1.1.** *Let  $K$  be an imaginary quadratic field,  $\Delta_K$  its discriminant,  $h_K$  its class number,  $\omega_K$  the number of units in  $\mathcal{O}_K$ . For  $i \in \{1, \dots, 4\}$ , let  $U_i$  be the complement of the lines in the del Pezzo surface  $S_i \subset \mathbb{P}_K^4$  defined by (1.2)–(1.5). For  $B \geq 3$ , we have*

$$N_{U_i, H}(B) = c_{S_i, H} B(\log B)^5 + O(B(\log B)^4 \log \log B),$$

with

$$c_{S_i, H} := \alpha(\tilde{S}_i) \cdot \frac{(2\pi)^6 h_K^6}{\Delta_K^4 \omega_K^6} \cdot \theta_0 \cdot \omega_\infty(\tilde{S}_i).$$

Here,

$$\begin{aligned} \alpha(\tilde{S}_1) &:= \frac{1}{8640}, & \alpha(\tilde{S}_2) &:= \frac{1}{21600}, & \alpha(\tilde{S}_3) &:= \frac{1}{34560}, & \alpha(\tilde{S}_4) &:= \frac{1}{345600}, \\ \theta_0 &:= \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^6 \left(1 + \frac{6}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right), \end{aligned} \quad (1.7)$$

and

$$\begin{aligned} \omega_\infty(\tilde{S}_1) &:= \frac{12}{\pi} \int_{\|z_0z_1(z_0+z_2)\|_\infty, \|z_1^3\|_\infty, \|z_1^2(z_0+z_2)\|_\infty, \|z_1z_2(z_0+z_2)\|_\infty, \|z_0z_2(z_0+z_2)\|_\infty \leq 1} dz_0 dz_1 dz_2, \\ \omega_\infty(\tilde{S}_2) &:= \frac{12}{\pi} \int_{\|z_0^3\|_\infty, \|z_0z_2z_3\|_\infty, \|z_0^2z_2\|_\infty, \|z_0^2z_3\|_\infty, \|z_3(z_2^2+z_0z_3)\|_\infty \leq 1} dz_0 dz_1 dz_2, \\ \omega_\infty(\tilde{S}_3) &:= \frac{12}{\pi} \int_{\|z_0z_1^2\|_\infty, \|z_1^3\|_\infty, \|z_1^2z_2\|_\infty, \|z_1(z_0z_1+z_2^2)\|_\infty, \|z_0(z_0z_1+z_2^2)\|_\infty \leq 1} dz_0 dz_1 dz_2, \\ \omega_\infty(\tilde{S}_4) &:= \frac{12}{\pi} \int_{\|z_0^3\|_\infty, \|z_0z_1^2\|_\infty, \|z_0^2z_1\|_\infty, \|z_0^2z_2\|_\infty, \|z_0z_2^2+z_1^3\|_\infty \leq 1} dz_0 dz_1 dz_2. \end{aligned}$$

We note that Manin's conjecture for  $S_4$  is implied by [CLT02] over arbitrary number fields, since  $S_4$  is an equivariant compactification of  $\mathbb{G}_a^2$ . On the other hand,  $S_0, \dots, S_3$  are neither toric nor equivariant compactifications of  $\mathbb{G}_a^2$  [DL10], so that [BT98a, CLT02] do not apply. Finally,  $S_1$  and  $S_3$  (but not  $S_0$ ,  $S_2$ ,  $S_4$ ) are equivariant compactifications of some semidirect products  $\mathbb{G}_a \rtimes \mathbb{G}_m$  [DL12],

so similar methods as in [BT98a, CLT02] may apply to them, but this has been worked out only over  $\mathbb{Q}$  and with further restrictions in [TT12].

**1.2. Methods.** The general strategy in our proofs of Theorem 1.1 for  $S_1, \dots, S_4$  is the one proposed in [DF13]:

In a first step, the rational points  $S_i(K)$  are parameterized by integral points on universal torsors over  $S_i$ , satisfying certain *height conditions* and *coprimality conditions*, following the strategy from [DF13, Section 4]. Since the Cox rings of all minimal desingularizations  $\tilde{S}_i$  have only one relation [Der06], the universal torsors are open subsets of hypersurfaces in  $\mathbb{A}_K^9$ , with coordinates  $(\eta_1, \dots, \eta_9)$  and one relation, the *torsor equation*.

In the second step, we approximate the number of these integral points on universal torsors subject to height and coprimality conditions by an integral. In all cases  $\eta_9$  appears linearly in the torsor equation, so it is uniquely defined by  $\eta_1, \dots, \eta_8$ . We first count pairs  $(\eta_8, \eta_9)$  for given  $(\eta_1, \dots, \eta_7)$  using the method from [DF13, Section 5] and then sum the result over another variable using the results from [DF13, Section 6]. The summations over the remaining variables are handled in all cases by a direct application of the results of [DF13, Section 7].

In a third and final step, we show that the integrals from the second step satisfy the asymptotic formulas from Theorem 1.1. Here, the shape of the effective cone of  $\tilde{S}_i$  is crucial; after all, the volume of its dual intersected with a certain hyperplane appears as  $\alpha(\tilde{S}_i)$  in Peyre's refinement [Pey95] of Manin's conjecture.

Though the proofs for  $S_0, \dots, S_4$  have many features in common, each case has its own difficulties.

In the case of  $S_0$ , the first step is mostly covered by our general results from [DF13], whereas the second step requires dichotomies with different orders of summation according to the relative size of the variables.

The first step in the case of  $S_1$  is mostly covered by the general results as well, but the second summation in the second step requires additional effort in order to obtain sufficiently good error terms.

In the case of  $S_2$ , parts of the first step need to be treated individually, and the second summation in the second step is more complicated, since  $\eta_8$  does not appear linearly in the torsor equation. Additionally, the second summation requires a dichotomy similarly as in the case of  $S_0$ , in order to handle the error terms.

The case of  $S_3$  is probably the most simple one. Parts of the first step need to be treated individually, but the summations in the second step go through without additional tricks, so it just remains to bound the error terms.

Finally, in the case of  $S_4$ , parts of the first step need to be treated individually, and the second summation in the second step is slightly more complicated, since  $\eta_8$  does not appear linearly in the torsor equation.

**1.3. Notation.** Throughout this article, we use the notation introduced in [DF13, Section 1.4]. In particular,  $\mathcal{C}$  denotes a fixed system of integral representatives for the ideal classes of the ring of integers  $\mathcal{O}_K$ . Moreover,  $\mathfrak{p}$  always denotes a nonzero prime ideal of  $\mathcal{O}_K$ , and products indexed by  $\mathfrak{p}$  are understood to run over all such prime ideals. We say that  $x \in K$  is *defined* (resp. *invertible*) modulo an ideal  $\mathfrak{a}$  of  $\mathcal{O}_K$ , if  $v_{\mathfrak{p}}(x) \geq 0$  (resp.  $v_{\mathfrak{p}}(x) = 0$ ) for all  $\mathfrak{p} \mid \mathfrak{a}$ , where  $v_{\mathfrak{p}}$  is the usual  $\mathfrak{p}$ -adic valuation. For  $x, y$  defined modulo  $\mathfrak{a}$ , we write  $x \equiv_{\mathfrak{a}} y$  if  $v_{\mathfrak{p}}(x - y) \geq v_{\mathfrak{p}}(\mathfrak{a})$  for all  $\mathfrak{p} \mid \mathfrak{a}$ .

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## 2. THE QUARTIC DEL PEZZO SURFACE OF TYPE $\mathbf{A}_3 + \mathbf{A}_1$

**2.1. Passage to a universal torsor.** Up to a permutation of the indices, we use the notation of [Der06].

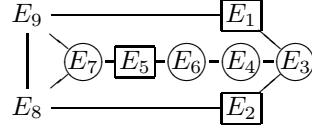


FIGURE 1. Configuration of curves on  $\tilde{S}_1$

For any given  $\mathbf{C} = (C_0, \dots, C_5) \in \mathcal{C}^6$ , we define  $u_{\mathbf{C}} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_5^{-1})$  and

$$\begin{aligned} \mathcal{O}_1 &:= C_5 & \mathcal{O}_2 &:= C_4 & \mathcal{O}_3 &:= C_0 C_1^{-1} C_4^{-1} C_5^{-1} \\ \mathcal{O}_4 &:= C_1 C_2^{-1} & \mathcal{O}_5 &:= C_3 & \mathcal{O}_6 &:= C_2 C_3^{-1} \\ \mathcal{O}_7 &:= C_0 C_1^{-1} C_2^{-1} C_3^{-1} & \mathcal{O}_8 &:= C_0 C_4^{-1} & \mathcal{O}_9 &:= C_0 C_5^{-1}. \end{aligned}$$

Let

$$\mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0}, & j \in \{1, \dots, 7\}, \\ \mathcal{O}_j, & j \in \{8, 9\}. \end{cases}$$

For  $\eta_j \in \mathcal{O}_j$ , let

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

For  $B \geq 0$ , let  $\mathcal{R}(B)$  be the set of all  $(\eta_1, \dots, \eta_8) \in \mathbb{C}^8$  with  $\eta_1 \neq 0$  and

$$\|\eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \eta_8\|_{\infty} \leq B, \quad (2.1)$$

$$\|\eta_1^2 \eta_2^2 \eta_3^3 \eta_4^2 \eta_6\|_{\infty} \leq B, \quad (2.2)$$

$$\|\eta_1 \eta_2 \eta_3^2 \eta_4^2 \eta_5^2 \eta_6^2 \eta_7\|_{\infty} \leq B, \quad (2.3)$$

$$\|\eta_3 \eta_4 \eta_5 \eta_6 \eta_7 (\eta_4 \eta_5^3 \eta_6^2 \eta_7 + \eta_2 \eta_8)\|_{\infty} \leq B, \quad (2.4)$$

$$\left\| \frac{\eta_2 \eta_7 \eta_8^2 + \eta_4 \eta_5^3 \eta_6^2 \eta_7^2 \eta_8}{\eta_1} \right\|_{\infty} \leq B. \quad (2.5)$$

We observe for future reference that (2.1) and (2.4) imply the condition

$$\|\eta_3 \eta_4^2 \eta_5^4 \eta_6^3 \eta_7^2\|_{\infty} \leq 4B. \quad (2.6)$$

Let  $M_{\mathbf{C}}(B)$  be the set of all

$$(\eta_1, \dots, \eta_9) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}$$

that satisfy the *height conditions*

$$(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B),$$

the *torsor equation*

$$\eta_4 \eta_5^3 \eta_6^2 \eta_7 + \eta_2 \eta_8 + \eta_1 \eta_9 = 0, \quad (2.7)$$

and the *coprimality conditions*

$$I_j + I_k = \mathcal{O}_K \text{ for all distinct nonadjacent vertices } E_j, E_k \text{ in Figure 1.} \quad (2.8)$$

**Lemma 2.1.** *We have*

$$N_{U_1, H}(B) = \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} |M_{\mathbf{C}}(B)|.$$

*Proof.* We observe that the statement of our lemma is a specialization of [DF13, Claim 4.1]. We prove it using the strategy from [DF13, Section 4] based on the construction of the minimal desingularization  $\pi : \tilde{S}_1 \rightarrow S_1$  by the following sequence of blow-ups: Starting with the curves  $E_8^{(0)} := \{y_0 = 0\}$ ,  $E_3^{(0)} := \{y_1 = 0\}$ ,  $E_9^{(0)} := \{y_2 = 0\}$ ,  $E_7^{(0)} := \{-y_0 - y_2 = 0\}$  in  $\mathbb{P}_K^2$ , we

- (1) blow up  $E_3^{(0)} \cap E_7^{(0)}$ , giving  $E_4^{(1)}$ ,
- (2) blow up  $E_4^{(1)} \cap E_7^{(1)}$ , giving  $E_6^{(2)}$ ,
- (3) blow up  $E_6^{(2)} \cap E_7^{(2)}$ , giving  $E_5^{(3)}$ ,
- (4) blow up  $E_3^{(3)} \cap E_8^{(3)}$ , giving  $E_2^{(4)}$ ,
- (5) blow up  $E_3^{(4)} \cap E_9^{(4)}$ , giving  $E_1^{(5)}$ .

With the inverse  $\pi \circ \rho^{-1} : \mathbb{P}_K^2 \dashrightarrow S_1$  of the projection  $\phi = \rho \circ \pi^{-1} : S_1 \dashrightarrow \mathbb{P}_K^2$ ,  $(x_0 : \dots : x_4) \mapsto (x_0 : x_2 : x_3)$  given by

$$\psi((y_0 : y_1 : y_2)) = (y_0 y_1 (y_0 + y_2) : -y_1^3 : y_1^2 (y_0 + y_2) : y_1 y_2 (y_0 + y_2) : y_0 y_2 (y_0 + y_2)) \quad (2.9)$$

and the map  $\Psi$  from [DF13, Claim 4.2] sending  $(\eta_1, \dots, \eta_9)$  to

$$(\eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \eta_8, -\eta_1^2 \eta_2^2 \eta_3^3 \eta_4^2 \eta_6, \eta_1 \eta_2 \eta_3^2 \eta_4^2 \eta_5^2 \eta_6^2 \eta_7, \eta_1 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \eta_9, \eta_7 \eta_8 \eta_9),$$

we can proceed exactly as in the proof of [DF13, Lemma 9.1].  $\square$

## 2.2. Summations.

### 2.2.1. The first summation over $\eta_8$ with dependent $\eta_9$ .

**Lemma 2.2.** *Write  $\boldsymbol{\eta}' := (\eta_1, \dots, \eta_7)$  and  $\mathbf{I}' := (I_1, \dots, I_7)$ . For  $B > 0$ ,  $\mathbf{C} \in \mathcal{C}^6$ , we have*

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{7*}} \theta_8(\mathbf{I}') V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_7; B) + O_{\mathbf{C}}(B(\log B)^2),$$

where

$$V_8(t_1, \dots, t_7; B) := \frac{1}{t_1} \int_{h(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8; B) \leq 1} d\eta_8,$$

with a complex variable  $\eta_8$ , and where

$$\theta_8(\mathbf{I}') := \prod_{\mathfrak{p}} \theta_{8,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}')),$$

with  $J_{\mathfrak{p}}(\mathbf{I}') := \{j \in \{1, \dots, 7\} : \mathfrak{p} \mid I_j\}$  and

$$\theta_{8,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{1\}, \{2\}, \{7\}, \\ 1 - \frac{1}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{4\}, \{5\}, \{6\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 6\}, \{5, 6\}, \{5, 7\}, \\ 1 - \frac{2}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{3\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By [DF13, Lemma 3.2], the set  $\mathcal{R}(\boldsymbol{\eta}', u_{\mathbf{C}} B)$  of all  $\eta_8 \in \mathbb{C}$  with  $(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B)$  has class  $m$ , with an absolute constant  $m$ . Moreover, by [DF13, Lemma 3.4, (1)] applied to (2.5), this set is contained in the union of at most 2 balls of radius

$$R(\boldsymbol{\eta}'; u_{\mathbf{C}} B) := (u_{\mathbf{C}} B \|\eta_1 \eta_2^{-1} \eta_7^{-1}\|_{\infty})^{1/4} \ll_{\mathbf{C}} (B \mathfrak{N}(I_1 I_2^{-1} I_7^{-1}))^{1/4}$$

We apply [DF13, Proposition 5.3] with  $(A_1, A_2, A_3, A_0) := (4, 6, 5, 7)$ ,  $(B_1, B_0) := (2, 8)$ ,  $(C_1, C_0) := (1, 9)$ ,  $D := 3$ , and  $u_{\mathbf{C}} B$  instead of  $B$ . (Moreover, we choose  $\Pi_1$  and  $\Pi_2$  as in [DF13, Remark 5.2].)

Similarly as in [DF13, Lemma 9.2], we see that the resulting main term is the one given in the lemma. The error term from [DF13, Proposition 5.3] is

$$\ll \sum_{\boldsymbol{\eta}', (2.10)} 2^{\omega_K(I_3) + \omega_K(I_3 I_4 I_5 I_6)} \left( \frac{R(\boldsymbol{\eta}'; u_{\mathbf{C}} B)}{\mathfrak{N}(I_1)^{1/2}} + 1 \right),$$

where, using (2.3) and the definitions of  $u_{\mathbf{C}}$  and the  $\mathcal{O}_j$ , the sum runs over all  $\boldsymbol{\eta}'$  with

$$\mathfrak{N}(I_1 I_2 I_3^2 I_4^2 I_5^2 I_6^2 I_7) \leq B. \quad (2.10)$$

Since  $|\mathcal{O}_K^\times| < \infty$ , we can sum over the  $I_j$  instead of the  $\eta_j$ , which then run over all nonzero ideals of  $\mathcal{O}_K$  with (2.10), so the error term is bounded by

$$\begin{aligned} &\ll_{\mathbf{C}} \sum_{\mathbf{I}', (2.10)} 2^{\omega_K(I_3) + \omega_K(I_3 I_4 I_5 I_6)} \left( \frac{B^{1/4}}{\mathfrak{N}I_1^{1/4} NI_2^{1/4} \mathfrak{N}I_7^{1/4}} + 1 \right) \\ &\ll \sum_{\substack{I_1, \dots, I_6 \\ \mathfrak{N}I_j \leq B}} \left( \frac{2^{\omega_K(I_3) + \omega_K(I_3 I_4 I_5 I_6)} B}{\mathfrak{N}I_1 \mathfrak{N}I_2 \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4^{3/2} \mathfrak{N}I_5^{3/2} \mathfrak{N}I_6^{3/2}} + \frac{2^{\omega_K(I_3) + \omega_K(I_3 I_4 I_5 I_6)} B}{\mathfrak{N}I_1 \mathfrak{N}I_2 \mathfrak{N}I_3^2 \mathfrak{N}I_4^2 \mathfrak{N}I_5^2 \mathfrak{N}I_6^2} \right) \\ &\ll B(\log B)^2 + B(\log B)^2 \ll B(\log B)^2. \end{aligned}$$

□

### 2.2.2. The second summation over $\eta_7$ .

**Lemma 2.3.** *Write  $\boldsymbol{\eta}'' := (\eta_1, \dots, \eta_6)$ . For  $B \geq 3$ ,  $\mathbf{C} \in \mathcal{C}^6$ , we have*

$$\begin{aligned} |M_{\mathbf{C}}(B)| &= \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}} \mathcal{A}(\theta_8(\mathbf{I}'), I_7) V_7(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) \\ &\quad + O_{\mathbf{C}}(B(\log B)^4 \log \log B). \end{aligned}$$

Here, for  $t_1, \dots, t_6 \geq 1$ ,

$$V_7(t_1, \dots, t_6; B) := \frac{\pi}{t_1} \int_{\substack{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B) \\ t_7 \geq 1}} dt_7 d\eta_8,$$

with a real variable  $t_7$  and a complex variable  $\eta_8$ .

*Proof.* Following the strategy described in [DF13, Section 6] in the case  $b_0 = 1$ , we write

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}} \sum_{\eta_7 \in \mathcal{O}_{7*}} \vartheta(I_7) g(\mathfrak{N}I_7) + O_{\mathbf{C}}(B(\log B)^2), \quad (2.11)$$

where  $\vartheta(\mathfrak{a}) := \theta_8(I_1, \dots, I_6, \mathfrak{a})$  and  $g(t) := V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6, t; B)$ . The conditions (2.2) and (2.6) imply that  $g(t) = 0$  unless

$$\mathfrak{N}I_1^2 \mathfrak{N}I_2^2 \mathfrak{N}I_3^3 \mathfrak{N}I_4^2 \mathfrak{N}I_6 \leq B \quad \text{and} \quad t \leq t_2 := \left( \frac{4B}{\mathfrak{N}I_3 \mathfrak{N}I_4^2 \mathfrak{N}I_5^4 \mathfrak{N}I_6^3} \right)^{1/2}. \quad (2.12)$$

Moreover, applying [DF13, Lemma 3.4, (2)] to (2.5), we see that

$$\begin{aligned} g(t) &\ll \frac{1}{\mathfrak{N}I_1} \cdot \left( \frac{\mathfrak{N}I_1 B}{\mathfrak{N}I_2 t} \right)^{1/2} \\ &= \frac{B}{\mathfrak{N}I_1 \dots \mathfrak{N}I_6 t} \left( \frac{B}{\mathfrak{N}I_1^2 \mathfrak{N}I_2^2 \mathfrak{N}I_3^3 \mathfrak{N}I_4^2 \mathfrak{N}I_6} \right)^{-1/4} \left( \frac{B}{\mathfrak{N}I_3 \mathfrak{N}I_4^2 \mathfrak{N}I_5^4 \mathfrak{N}I_6^3 t^2} \right)^{-1/4}. \end{aligned}$$

In particular, we always have  $g(t) \ll B/(\mathfrak{N}I_1 \dots \mathfrak{N}I_6 t)$ .

By [DF13, Lemma 5.4, Lemma 2.2],  $\vartheta$  satisfies the condition [DF13, (6.1)] with  $C = 0$  and  $c_{\vartheta} = 2^{\omega(I_1 \dots I_6)}$ .

Let  $t_1 := (\log B)^{14}$ . A straightforward application of [DF13, Proposition 6.1] would not yield sufficiently good error terms, so, using a strategy as in the proof of [DF13, Proposition 7.2], we split the sum over  $\eta_7$  into the two cases  $\mathfrak{N}I_7 \leq t_1$  and  $\mathfrak{N}I_7 > t_1$ .

Let us start with the second case. We may assume that  $t_2 \geq t_1$ . Using [DF13, Proposition 6.1] with the upper bound  $g(t) \ll B/(\mathfrak{N}I_1 \dots \mathfrak{N}I_6 t)$ , we see that

$$\begin{aligned} \sum_{\substack{\eta_7 \in \mathcal{O}_{7*} \\ \mathfrak{N}I_7 > t_1}} \vartheta(I_7) g(\mathfrak{N}I_7) &= \frac{2\pi}{\sqrt{|\Delta_K|}} \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathcal{O}_K) \int_{t \geq t_1} g(t) dt \\ &\quad + O\left(\frac{2^{\omega_K(I_1 \dots I_4 I_6)} B}{\mathfrak{N}I_1 \dots \mathfrak{N}I_6} t_1^{-1/2}\right). \end{aligned}$$

When summing the error term over the remaining variables, we may sum over all  $\mathbf{I}''$  with  $\mathfrak{N}I_j \leq B$ , so the error term is

$$\ll t_1^{-1/2} \sum_{\mathbf{I}''} \frac{2^{\omega_K(I_1 \dots I_4 I_6)} B}{\mathfrak{N}I_1 \dots \mathfrak{N}I_6} \ll (\log B)^{-7} B (\log B)^{11} = B (\log B)^4.$$

Now let us consider the sum over all  $\eta_7$  with  $\mathfrak{N}I_7 \leq t_1$ . Since  $0 \leq \vartheta(I_7) \leq 1$ , we obtain an upper bound

$$\begin{aligned} &\sum_{\mathbf{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}} \sum_{\substack{\eta_7 \in \mathcal{O}_{7*} \\ \mathfrak{N}I_7 \leq t_1}} \vartheta(I_7) g(\mathfrak{N}I_7) \\ &\ll \sum_{\substack{\mathbf{I}'', I_7 \\ (2.12) \text{ with } t = \mathfrak{N}I_7 \\ \mathfrak{N}I_7 \leq t_1}} \frac{B}{\mathfrak{N}I_1 \dots \mathfrak{N}I_6 \mathfrak{N}I_7} \left( \frac{B}{\mathfrak{N}I_1^2 \mathfrak{N}I_2^2 \mathfrak{N}I_3^3 \mathfrak{N}I_4^2 \mathfrak{N}I_6} \right)^{-\frac{1}{4}} \left( \frac{B}{\mathfrak{N}I_3 \mathfrak{N}I_4^2 \mathfrak{N}I_5^4 \mathfrak{N}I_6^3 \mathfrak{N}I_7^2} \right)^{-\frac{1}{4}} \\ &\ll \sum_{\substack{I_2, \dots, I_7 \\ (2.12) \text{ with } t = \mathfrak{N}I_7 \\ \mathfrak{N}I_7 \leq t_1}} \frac{B}{\mathfrak{N}I_2 \dots \mathfrak{N}I_7} \left( \frac{B}{\mathfrak{N}I_3 \mathfrak{N}I_4^2 \mathfrak{N}I_5^4 \mathfrak{N}I_6^3 \mathfrak{N}I_7^2} \right)^{-\frac{1}{4}} \\ &\ll \sum_{\substack{I_2, I_3, I_4, I_6, I_7 \\ \mathfrak{N}I_j \leq B, \mathfrak{N}I_7 \leq t_1}} \frac{B}{\mathfrak{N}I_2 \mathfrak{N}I_3 \mathfrak{N}I_4 \mathfrak{N}I_6 \mathfrak{N}I_7} \ll B (\log B)^4 \log t_1 \ll B (\log B)^4 \log \log B. \end{aligned}$$

Our proof is finished once we see that

$$\sum_{\mathbf{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}} \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}) \int_1^{t_1} g(t) dt \ll B (\log B)^4 \log \log B.$$

This follows from an analogous computation as above with the integral over  $t$  instead of the sum over  $I_7$ , and using that  $0 \leq \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}) \leq 1$ .  $\square$

**Lemma 2.4.** *If  $\mathbf{I}''$  runs over all six-tuples  $(I_1, \dots, I_6)$  of nonzero ideals of  $\mathcal{O}_K$  then we have*

$$\begin{aligned} N_{U_1, H}(B) &= \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta_8(\mathbf{I}'', I_7), I_7) V_7(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) \\ &\quad + O(B (\log B)^4 \log \log B). \end{aligned}$$

*Proof.* This is entirely analogous to [DF13, Lemma 9.4].  $\square$

2.2.3. *The remaining summations.*

**Lemma 2.5.** *We have*

$$N_{U_1, H}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^8 \left( \frac{h_K}{\omega_K} \right)^6 \theta_0 V_0(B) + O(B(\log B)^4 \log \log B),$$

where  $\theta_0$  is as in (1.7) and

$$V_0(B) := \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geq 1}} \frac{1}{\|\eta_1\|_\infty} d\eta_1 \cdots d\eta_8,$$

with complex variables  $\eta_1, \dots, \eta_8$ .

*Proof.* By [DF13, Lemma 3.4, (6)], applied to the (2.5), we have

$$V_7(t_1, \dots, t_6; B) \ll \frac{B^{2/3}}{t_1^{1/3} t_2^{1/3} t_4^{1/3} t_5 t_6^{2/3}} = \frac{B}{t_1 \cdots t_6} \left( \frac{B}{t_1^2 t_2^2 t_3^3 t_4^2 t_6} \right)^{-1/3}.$$

We apply [DF13, Proposition 7.3] with  $r = 5$  and use polar coordinates, similarly to [DF13, Lemma 9.5, Lemma 9.9].  $\square$

2.3. **Proof of Theorem 1.1 for  $S_1$ .** We will use the conditions

$$\|\eta_1^2 \eta_2^2 \eta_4^2 \eta_6\|_\infty \leq B \text{ and} \quad (2.13)$$

$$\|\eta_1^2 \eta_2^2 \eta_4^2 \eta_6\|_\infty \leq B \text{ and } \|\eta_1^{-1} \eta_2^{-1} \eta_4^2 \eta_5^6 \eta_6^4\|_\infty \leq B. \quad (2.14)$$

**Lemma 2.6.** *Let  $\alpha(\tilde{S}_1)$ ,  $\omega_\infty(\tilde{S}_1)$  be as in Theorem 1.1, let  $\mathcal{R}(B)$  be as in (2.1)–(2.5), and define*

$$V'_0(B) := \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1}} \frac{1}{\|\eta_1\|_\infty} d\eta_1 \cdots d\eta_8, \quad (2.14)$$

with complex variables  $\eta_1, \dots, \eta_8$ . Then

$$\pi^6 \alpha(\tilde{S}_1) \omega_\infty(\tilde{S}_1) B (\log B)^5 = 4V'_0(B). \quad (2.15)$$

*Proof.* We use the following substitutions on  $\omega_\infty(\tilde{S}_1)$ : Let  $\eta_1, \eta_2, \eta_4, \eta_5, \eta_6 \in \mathbb{C} \setminus \{0\}$  and  $B > 0$ . Let  $\eta_3, \eta_7, \eta_8$  be complex variables. With  $l := (B \|\eta_1 \eta_2 \eta_4 \eta_5^3 \eta_6^2\|_\infty)^{1/2}$ , we apply the coordinate transformation  $z_0 = l^{-1/3} \eta_2 \cdot \eta_8$ ,  $z_1 = l^{-1/3} \eta_1 \eta_2 \eta_4 \eta_5 \eta_6 \cdot \eta_3$ ,  $z_2 = l^{-1/3} (-\eta_2 \cdot \eta_8 - \eta_4 \eta_5^3 \eta_6^2 \cdot \eta_7)$ , of Jacobi determinant

$$\frac{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}{B} \frac{1}{\|\eta_1\|_\infty}, \quad (2.16)$$

and obtain

$$\omega_\infty(\tilde{S}_1) = \frac{12}{\pi} \frac{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}{B} \int_{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B)} \frac{1}{\|\eta_1\|_\infty} d\eta_3 d\eta_7 d\eta_8. \quad (2.17)$$

The negative curves  $[E_1], \dots, [E_7]$  generate the effective cone of  $\tilde{S}_1$ . We have  $[-K_{\tilde{S}_1}] = [2E_1 + 2E_2 + 3E_3 + 2E_4 + E_6]$  and  $[E_7] = [E_1 + E_2 + E_3 - 2E_5 - E_6]$ . Hence, [DF13, Lemma 8.1] (with the roles of  $\eta_3$  and  $\eta_6$  exchanged) gives

$$\alpha(\tilde{S}_1) (\log B)^5 = \frac{1}{3\pi^5} \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1}} \frac{d\eta_1 d\eta_2 d\eta_4 d\eta_5 d\eta_6}{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}. \quad (2.18)$$

The lemma follows by substituting (2.17) and (2.18) in (2.15).  $\square$

To finish our proof, we compare  $V_0(B)$  from Lemma 2.5 with  $V'_0(B)$  defined in Lemma 2.6. Let

$$\begin{aligned}\mathcal{D}_0(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geq 1\}, \\ \mathcal{D}_1(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geq 1, (2.13)\}, \\ \mathcal{D}_2(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geq 1, (2.14)\}, \\ \mathcal{D}_3(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geq 1, (2.14)\}, \\ \mathcal{D}_4(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1, (2.14)\}.\end{aligned}$$

Moreover, let

$$V_i(B) := \int_{\mathcal{D}_i(B)} \frac{d\eta_1 \cdots d\eta_8}{\|\eta_1\|_\infty}.$$

Then  $V_0(B)$  is as in Lemma 2.5 and  $V_4(B) = V'_0(B)$ . We show that, for  $1 \leq i \leq 4$ ,  $V_i(B) - V_{i-1}(B) = O(B(\log B)^4)$ . This holds for  $i = 1$ , since, by (2.2) and  $\|\eta_3\|_\infty \geq 1$ , we have  $\mathcal{D}_1(B) = \mathcal{D}_0(B)$ .

Moreover, using [DF13, Lemma 3.4, (2)] and (2.5) to bound the integral over  $\eta_8$ , we have

$$V_2(B) - V_1(B) \ll \int_{\substack{1 \leq \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \leq B \\ \|\eta_1^{-1} \eta_2^{-1} \eta_4^2 \eta_5^6 \eta_6^4\|_\infty > B}} \frac{B^{1/2}}{\|\eta_1 \eta_2 \eta_7\|_\infty^{1/2}} d\eta_1 \cdots d\eta_7 \ll B(\log B)^4.$$

Moreover,

$$V_3(B) - V_2(B) \ll \int_{\substack{\|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geq 1 \\ \|\eta_7\|_\infty < 1, (2.2), (2.14)}} \frac{B^{1/2}}{\|\eta_1 \eta_2 \eta_7\|_\infty^{1/2}} d\eta_1 \cdots d\eta_7 \ll B(\log B)^4.$$

Finally, using [DF13, Lemma 3.4, (4)] and (2.5) to bound the integral over  $\eta_7$ , we have

$$V_4(B) - V_3(B) \ll \int_{\substack{\|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1 \\ \|\eta_3\|_\infty < 1, (2.13)}} \frac{B^{2/3}}{\|\eta_1 \eta_2 \eta_4 \eta_5^3 \eta_6^2\|_\infty^{1/3}} d\eta_1 \cdots d\eta_6 \ll B(\log B)^4.$$

Using Lemma 2.5 and Lemma 2.6, this shows Theorem 1.1 for  $S_1$ .

### 3. THE QUARTIC DEL PEZZO SURFACE OF TYPE $\mathbf{A}_4$

**3.1. Passage to a universal torsor.** We use the notation of [Der06], except that we swap  $\eta_8$  and  $\eta_9$ .

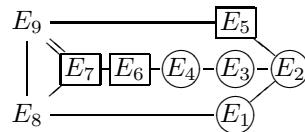


FIGURE 2. Configuration of curves on  $\tilde{S}_2$

For any given  $\mathbf{C} = (C_0, \dots, C_5) \in \mathcal{C}^6$ , we define  $u_{\mathbf{C}} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_5^{-1})$  and

$$\begin{aligned}\mathcal{O}_1 &:= C_3 C_4^{-1} & \mathcal{O}_2 &:= C_4 C_5^{-1} & \mathcal{O}_3 &:= C_0 C_1^{-1} C_3^{-1} C_4^{-1} \\ \mathcal{O}_4 &:= C_1 C_2^{-1} & \mathcal{O}_5 &:= C_5 & \mathcal{O}_6 &:= C_2 \\ \mathcal{O}_7 &:= C_0 C_1^{-1} C_2^{-1} & \mathcal{O}_8 &:= C_0 C_3^{-1} & \mathcal{O}_9 &:= C_0^2 C_3^{-1} C_4^{-1} C_5^{-1}.\end{aligned}$$

Let

$$\mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0}, & j \in \{1, \dots, 7\}, \\ \mathcal{O}_j, & j \in \{8, 9\}. \end{cases}$$

For  $\eta_j \in \mathcal{O}_j$ , let

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

For  $B \geq 0$ , let  $\mathcal{R}(B)$  be the set of all  $(\eta_1, \dots, \eta_8) \in \mathbb{C}^8$  with  $\eta_5 \neq 0$  and

$$\|\eta_1^2 \eta_2^4 \eta_3^3 \eta_4^2 \eta_5^3 \eta_6\|_\infty \leq B, \quad (3.1)$$

$$\|\eta_1 \eta_2 \eta_3 \eta_4 \eta_6 \eta_7 \eta_8\|_\infty \leq B, \quad (3.2)$$

$$\|\eta_1^2 \eta_2^3 \eta_3^2 \eta_4 \eta_5^2 \eta_8\|_\infty \leq B, \quad (3.3)$$

$$\|\eta_1 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5^2 \eta_6^2 \eta_7\|_\infty \leq B, \quad (3.4)$$

$$\left\| \frac{\eta_1 \eta_7 \eta_8^2 + \eta_3 \eta_4^2 \eta_6^3 \eta_7^2}{\eta_5} \right\|_\infty \leq B, \quad (3.5)$$

and let  $M_{\mathbf{C}}(B)$  be the set of all

$$(\eta_1, \dots, \eta_9) \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{9*}$$

that satisfy the *height conditions*

$$(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B),$$

the *torsor equation*

$$\eta_3 \eta_4^2 \eta_6^3 \eta_7 + \eta_1 \eta_8^2 + \eta_5 \eta_9 = 0, \quad (3.6)$$

and the *coprimality conditions*

$$I_j + I_k = \mathcal{O}_K \text{ for all distinct nonadjacent vertices } E_j, E_k \text{ in Figure 2.} \quad (3.7)$$

**Lemma 3.1.** *We have*

$$N_{U_2, H}(B) = \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} |M_{\mathbf{C}}(B)|.$$

*Proof.* This is a specialization of [DF13, Claim 4.1] and we prove it using the strategy from [DF13, Section 4] with the data supplied in [Der06]. Starting with the curves  $E_3^{(0)} := \{y_0 = 0\}$ ,  $E_8^{(0)} := \{y_1 = 0\}$ ,  $E_7^{(0)} := \{y_2 = 0\}$ ,  $E_9^{(0)} := \{-y_0 y_2 - y_1^2 = 0\}$  in  $\mathbb{P}_K^2$ , we prove [DF13, Claim 4.2] for the following sequence of blow-ups:

- (1) blow up  $E_3^{(0)} \cap E_8^{(0)} \cap E_9^{(0)}$ , giving  $E_1^{(1)}$ ,
- (2) blow up  $E_1^{(1)} \cap E_3^{(1)} \cap E_9^{(1)}$ , giving  $E_2^{(2)}$ ,
- (3) blow up  $E_2^{(2)} \cap E_9^{(2)}$ , giving  $E_5^{(3)}$ ,
- (4) blow up  $E_3^{(3)} \cap E_7^{(3)}$ , giving  $E_4^{(4)}$ ,
- (5) blow up  $E_4^{(4)} \cap E_7^{(4)}$ , giving  $E_6^{(5)}$ .

The inverse  $\pi \circ \rho^{-1} : \mathbb{P}_K^2 \dashrightarrow S_2$  of the projection  $\phi = \rho \circ \pi^{-1} : S_2 \dashrightarrow \mathbb{P}_K^2$ ,  $(x_0 : \dots : x_4) \mapsto (x_0 : x_2 : x_3)$  given by

$$(y_0 : y_1 : y_2) \mapsto (y_0^3 : y_0 y_1 y_2 : y_0^2 y_1 : y_0^2 y_2 : -y_2(y_1^2 + y_0 y_2)), \quad (3.8)$$

and the map  $\Psi$  appearing in [DF13, Claim 4.2] sends  $(\eta_1, \dots, \eta_9)$  to

$$(\eta_1^2 \eta_2^4 \eta_3^3 \eta_4^2 \eta_5^3 \eta_6, \eta_1 \eta_2 \eta_3 \eta_4 \eta_6 \eta_7 \eta_8, \eta_1^2 \eta_2^3 \eta_3^2 \eta_4 \eta_5^2 \eta_8, \eta_1 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5^2 \eta_6^2 \eta_7, \eta_7 \eta_9).$$

As in the proof of [DF13, Lemma 9.1], we see that the hypotheses of [DF13, Lemma 4.3] are satisfied, so [DF13, Claim 4.2] holds in our situation for  $i = 0$ .

Note that [DF13, Lemma 4.4] applies in steps (3), (4), (5) of the above chain of blow-ups. In steps (1), (2), we are in the situation of [DF13, Remark 4.5], so that

we must derive some coprimality conditions using the torsor equation. We use the notation of [DF13, Lemma 4.4, Remark 4.5].

For (1), we start with the parameterization provided by [DF13, Lemma 4.3], consisting of  $(\eta'_3, \eta'_7, \eta'_8, \eta'_9)$  satisfying certain coprimality conditions and other conditions. Since  $\eta'_3 \neq 0$ , there is a unique  $C_1 \in \mathcal{C}$  such that  $[I'_3 + I'_8 + I'_9] = [C_1^{-1}]$ . We choose  $\eta''_1 \in C_1$  such that  $I''_1 = I'_3 + I'_8 + I'_9$ ; this is unique up to multiplication by  $\mathcal{O}_K^\times$ . We define  $\eta''_3 := \eta'_3/\eta''_1$ ,  $\eta''_8 := \eta'_8/\eta''_1$ ,  $\eta''_9 := \eta'_9/\eta''_1$  and  $\eta''_7 := \eta'_7$ . To show that  $(\eta''_1, \eta''_3, \eta''_7, \eta''_8, \eta''_9)$  lies in the set described in [DF13, Claim 4.2] for  $i = 1$ , everything is provided by the proof of [DF13, Lemma 4.4] except the coprimality conditions involving  $\eta''_1, \eta''_3, \eta''_8, \eta''_9$ . Considering the configuration of  $E_1^{(1)}, E_3^{(1)}, E_8^{(1)}, E_9^{(1)}$ , these are  $I''_3 + I''_8 = \mathcal{O}_K$  (which holds because  $I''_3 + I''_8 + I''_9 = \mathcal{O}_K$  by construction and because of the relation  $\eta''_3\eta''_7 + \eta''_1\eta''_8 + \eta''_9 = 0$ ) and  $I''_1 + I''_8 + I''_9 = \mathcal{O}_K$  (which holds because otherwise the relation would give non-triviality of  $I''_1 + I''_8 + I''_9 + I''_3 I''_7$  contradicting the previous condition or the condition  $I''_1 + I''_7 = \mathcal{O}_K$  provided by the proof of [DF13, Lemma 4.4]).

For (2), we replace  $''$  by  $'$  in the result of the previous step. We choose  $C_2 \in \mathcal{C}$  such that  $[I'_1 + I'_3 + I'_9] = [C_2^{-1}]$  and  $\eta''_2 \in C_2 = \mathcal{O}_2''$  such that  $I''_2 = I'_1 + I'_3 + I'_9$ . It remains to check the pairwise coprimality of  $I'_1, I'_3, I'_9$ . By construction,  $I''_1 + I''_3 + I''_9 = \mathcal{O}_K$ ; considering the torsor equation  $\eta''_3\eta''_7 + \eta''_1\eta''_8 + \eta''_9 = 0$  shows  $I''_1 + I''_3 = \mathcal{O}_K$  directly,  $I''_1 + I''_9 = \mathcal{O}_K$  using  $I''_1 + I''_7 = \mathcal{O}_K$ , and  $I''_3 + I''_9 = \mathcal{O}_K$  using  $I''_3 + I''_8 = \mathcal{O}_K$ .

Since steps (3), (4), (5) are covered by [DF13, Lemma 4.4], this shows [DF13, Claim 4.2]. We deduce [DF13, Claim 4.1] in the same way as in [DF13, Lemma 9.1].  $\square$

### 3.2. Summations.

**3.2.1. The first summation over  $\eta_8$  with dependent  $\eta_9$ .** Let  $\boldsymbol{\eta}' := (\eta_1, \dots, \eta_7)$  and  $\mathbf{I}' := (I_1, \dots, I_7)$ . Let  $\theta_0(\mathbf{I}') := \prod_{\mathfrak{p}} \theta_{0,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}'))$ , where  $J_{\mathfrak{p}}(\mathbf{I}') := \{j \in \{1, \dots, 7\} : \mathfrak{p} \mid I_j\}$  and

$$\theta_{0,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \\ & \text{or } J = \{1, 2\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 6\}, \{6, 7\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\theta_0(\mathbf{I}') = 1$  if and only if  $I_1, \dots, I_7$  satisfy the coprimality conditions from (3.7), and  $\theta_0(\mathbf{I}') = 0$  otherwise.

We apply [DF13, Proposition 5.3] with  $(A_1, A_2, A_3, A_0) := (3, 4, 6, 7)$ ,  $(B_1, B_0) := (1, 8)$ ,  $(C_1, C_0) := (5, 9)$ , and  $D := 2$ . For given  $\eta_2, \eta_5$ , we write

$$\eta_3\eta_4^2\eta_6^3\eta_7 = \eta_{A_0}^{a_0}\Pi(\boldsymbol{\eta}_A) = \Pi_1\Pi_2^2,$$

where  $\Pi_1, \Pi_2$  are chosen as follows: Let  $\mathfrak{A} = \mathfrak{A}(\eta_2, \eta_5)$  be a prime ideal not dividing  $I_2 I_5$  such that  $\mathfrak{A}\mathcal{O}_6^{-1}\mathcal{O}_8 = \mathfrak{A}C_0 C_2^{-1} C_3^{-1}$  is a principal fractional ideal  $t\mathcal{O}_K$ , for a suitable  $t = t(\eta_2, \eta_5) \in K^\times$ . Then we define  $\Pi_2 = \Pi_2(\eta_2, \eta_5) := \eta_6 t$  and  $\Pi_1 := \Pi_1(\eta_2, \eta_5) := \eta_3\eta_4^2\eta_6\eta_7 t^{-2}$ .

**Lemma 3.2.** *We have*

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{7*}} \theta_8(\boldsymbol{\eta}', \mathbf{C}) V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_7; B) + O_{\mathbf{C}}(B(\log B)^3),$$

where

$$V_8(t_1, \dots, t_7; B) := \frac{1}{t_5} \int_{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B)} d\eta_8.$$

Moreover,

$$\theta_8(\boldsymbol{\eta}', \mathbf{C}) := \sum_{\substack{\mathfrak{k}_c | I_2 \\ \mathfrak{k}_c + I_1 I_3 = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_c)}{\mathfrak{N}\mathfrak{k}_c} \tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_c) \sum_{\substack{\rho \mod \mathfrak{k}_c I_5 \\ \rho \mathcal{O}_K + \mathfrak{k}_c I_5 = \mathcal{O}_K \\ \rho^2 \equiv \mathfrak{k}_c I_5 \eta_6 \eta_7 A}} 1,$$

with

$$\tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_c) := \theta_0(\mathbf{I}') \frac{\phi_K^*(I_2 I_3 I_4 I_6)}{\phi_K^*(I_2 + \mathfrak{k}_c I_5)}.$$

Here,  $A := -\eta_3 \eta_4^2 / (t(\eta_2, \eta_5)^2 \eta_1)$ , and  $\eta_6 \eta_7 A$  is invertible modulo  $\mathfrak{k}_c I_5$  whenever  $\theta_0(\mathbf{I}') \neq 0$ .

*Proof.* It is clear that  $\theta_8(\boldsymbol{\eta}', \mathbf{C}) = \theta_1(\boldsymbol{\eta}')$  from [DF13, Proposition 5.3], and a simple argument as in the proof of [DF13, Lemma 9.2] shows that  $V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_7; B) = V_1(\boldsymbol{\eta}', u_{\mathbf{C}} B)$ . Hence, the main term is correct and it remains to bound the error term arising from [DF13, Proposition 5.3].

Similarly as in [DF13, Lemma 9.2], we see that the set  $\mathcal{R}(\boldsymbol{\eta}', B)$  of all  $\eta_8$  with  $(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B)$  is of bounded class and (using [DF13, Lemma 3.5, (1)] on (3.5)) contained in two balls of radius  $R(\boldsymbol{\eta}'; u_{\mathbf{C}} B) \ll_{\mathbf{C}} (B \mathfrak{N}I_5 \mathfrak{N}I_1^{-1} \mathfrak{N}I_7^{-1})^{1/4}$ .

The error term is

$$\ll \sum_{\boldsymbol{\eta}', (3.9)} 2^{\omega_K(I_2) + \omega_K(I_2 I_3 I_4 I_6) + \omega_K(I_2 I_5)} \left( \frac{R(\boldsymbol{\eta}'; u_{\mathbf{C}} B)}{\mathfrak{N}(I_5)^{1/2}} + 1 \right),$$

where, using (3.4), the sum runs over all  $\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{7*}$  with

$$\mathfrak{N}(I_1 I_2^2 I_3^2 I_4^2 I_5 I_6^2 I_7) \leq B. \quad (3.9)$$

Since  $|\mathcal{O}_K^\times| < \infty$ , we can sum over the  $I_j$  instead of the  $\eta_j$ , which then run over all nonzero ideals of  $\mathcal{O}_K$  with (3.9), and obtain

$$\begin{aligned} &\ll_{\mathbf{C}} \sum_{\mathbf{I}', (3.9)} 2^{\omega_K(I_2) + \omega_K(I_2 I_3 I_4 I_6) + \omega_K(I_2 I_5)} \left( \frac{B^{1/4}}{\mathfrak{N}I_1^{1/4} \mathfrak{N}I_5^{1/4} \mathfrak{N}I_7^{1/4}} + 1 \right) \\ &\ll \sum_{I_1, \dots, I_6} \left( \frac{2^{\omega_K(I_2) + \omega_K(I_2 I_3 I_4 I_6) + \omega_K(I_2 I_5)} B}{\mathfrak{N}I_1 \mathfrak{N}I_2^{3/2} \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4^{3/2} \mathfrak{N}I_5 \mathfrak{N}I_6^{3/2}} + \frac{2^{\omega_K(I_2) + \omega_K(I_2 I_3 I_4 I_6) + \omega_K(I_2 I_5)} B}{\mathfrak{N}I_1 \mathfrak{N}I_2^2 \mathfrak{N}I_3^2 \mathfrak{N}I_4^2 \mathfrak{N}I_5 \mathfrak{N}I_6^2} \right) \\ &\ll B(\log B)^3 + B(\log B)^3 \ll B(\log B)^3. \end{aligned} \quad \square$$

For the further summations, we define

$$\theta'_8(\mathbf{I}') := \sum_{\substack{\mathfrak{k}_c | I_2 \\ \mathfrak{k}_c + I_1 I_3 = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_c)}{\mathfrak{N}\mathfrak{k}_c} \tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_c)$$

and distinguish between two cases: Similarly to [BD09], let  $M_{\mathbf{C}}^{(86)}(B)$  be the main term in Lemma 3.2 with the additional condition  $\mathfrak{N}I_6 > \mathfrak{N}I_7$  on the  $\boldsymbol{\eta}'$ , and let  $M_{\mathbf{C}}^{(87)}(B)$  be the main term with the additional condition  $\mathfrak{N}I_6 \leq \mathfrak{N}I_7$ . Moreover, we define

$$N_{86}(B) := \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} M_{\mathbf{C}}^{(86)}(B)$$

and  $N_{87}(B)$  analogously, so

$$N_{U_2, H}(B) = N_{86}(B) + N_{87}(B) + O(B(\log B)^3). \quad (3.10)$$

3.2.2. *The second summation over  $\eta_6$  in  $M_{\mathbf{C}}^{(86)}(B)$ .*

**Lemma 3.3.** *Write  $\boldsymbol{\eta}'' := (\eta_1, \dots, \eta_5, \eta_7)$  and  $\mathcal{O}'' := \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{5*} \times \mathcal{O}_{7*}$ . We have*

$$M_{\mathbf{C}}^{(86)}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}''} \mathcal{A}(\theta'_8(\mathbf{I}'), I_6) V_{86}(\mathfrak{N}I_1, \dots, \mathfrak{N}I_5, \mathfrak{N}I_7; B) + O_{\mathbf{C}}(B(\log B)^4),$$

where, for  $t_1, \dots, t_5, t_7 \geq 1$ ,

$$V_{86}(t_1, \dots, t_5, t_7; B) := \frac{\pi}{t_5} \int_{\substack{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B) \\ t_6 > t_7}} dt_6 d\eta_8,$$

with a real variable  $t_6$  and a complex variable  $\eta_8$ .

*Proof.* We follow the strategy described in [DF13, Section 6] in the case  $b_0 \geq 2$ . We write

$$M_{\mathbf{C}}^{(86)}(B) = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \mathcal{O}''} \sum_{\substack{\mathfrak{k}_c | I_2 \\ \mathfrak{k}_c + I_1 I_3 = \mathcal{O}_K}} \frac{\mu(\mathfrak{k}_c)}{\mathfrak{N}\mathfrak{k}_c} \Sigma,$$

where

$$\Sigma := \sum_{\substack{\eta_6 \in \mathcal{O}_{6*} \\ \mathfrak{N}I_6 > \mathfrak{N}I_7}} \vartheta(I_6) \sum_{\substack{\rho \mod \mathfrak{k}_c I_5 = \mathcal{O}_K \\ \rho^2 \equiv \mathfrak{k}_c I_5 \eta_6 \eta_7 A}} g(\mathfrak{N}I_6),$$

with  $\vartheta(I_6) := \tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_c)$  and  $g(t) := V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_5, t, \mathfrak{N}I_7; B)$ .

By [DF13, Lemma 5.5, Lemma 2.2], the function  $\vartheta$  satisfies [DF13, (6.1)] with  $C := 0$ ,  $c_{\vartheta} := 2^{\omega_K(I_1 I_2 I_3 I_5)}$ . By (3.4), we have  $g(t) = 0$  whenever  $t > t_2 := B^{1/2}/(\mathfrak{N}I_1^{1/2} \mathfrak{N}I_2 \mathfrak{N}I_3 \mathfrak{N}I_4 \mathfrak{N}I_5^{1/2} \mathfrak{N}I_7^{1/2})$ , and, by Lemma [DF13, Lemma 3.5, (2)] applied to (3.5), we have  $g(t) \ll B^{1/2}/(\mathfrak{N}I_1^{1/2} \mathfrak{N}I_5^{1/2} \mathfrak{N}I_7^{1/2})$ . Using [DF13, Proposition 6.1], we obtain

$$\begin{aligned} \Sigma &= \frac{2\pi}{\sqrt{|\Delta_K|}} \phi_K^*(\mathfrak{k}_c I_5) \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathfrak{k}_c I_5) \int_{t \geq \mathfrak{N}I_7} g(t) dt \\ &\quad + O\left(\frac{2^{\omega_K(I_1 I_2 I_3 I_5)} B^{1/2}}{\mathfrak{N}I_1^{1/2} \mathfrak{N}I_5^{1/2} \mathfrak{N}I_7^{1/2}} \left( \frac{B^{1/4} \mathfrak{N}\mathfrak{k}_c^{1/2} \mathfrak{N}I_5^{1/4}}{\mathfrak{N}I_1^{1/4} \mathfrak{N}I_2^{1/2} \mathfrak{N}I_3^{1/2} \mathfrak{N}I_4^{1/2} \mathfrak{N}I_7^{1/4}} + \mathfrak{N}(\mathfrak{k}_c I_5) \log B \right)\right). \end{aligned}$$

Using [DF13, Lemma 6.3] we see that the main term in the lemma is correct.

For the error term, we may sum over  $\mathfrak{k}_c$  and over the ideals  $I_j$  instead of the  $\eta_j$ , since  $|\mathcal{O}_K^\times| < \infty$ . By (3.1) and our condition  $\mathfrak{N}I_6 > \mathfrak{N}I_7$  it suffices to sum over  $\mathfrak{k}_c$  and all  $(I_1, \dots, I_5, I_7)$  satisfying

$$\mathfrak{N}I_1^2 \mathfrak{N}I_2^4 \mathfrak{N}I_3^3 \mathfrak{N}I_4^2 \mathfrak{N}I_5^3 \mathfrak{N}I_7 \leq B. \quad (3.11)$$

Thus, the total error is bounded by

$$\begin{aligned} &\sum_{\substack{I_1, \dots, I_5, I_7 \\ (3.11)}} \left( \frac{2^{\omega_K(I_2) + \omega_K(I_1 I_2 I_3 I_5)} B^{3/4}}{\mathfrak{N}I_1^{3/4} \mathfrak{N}I_2^{1/2} \mathfrak{N}I_3^{1/2} \mathfrak{N}I_4^{1/2} \mathfrak{N}I_5^{1/4} \mathfrak{N}I_7^{3/4}} + \frac{2^{\omega_K(I_2) + \omega_K(I_1 I_2 I_3 I_5)} B^{1/2} \log B}{\mathfrak{N}I_1^{1/2} \mathfrak{N}I_5^{-1/2} \mathfrak{N}I_7^{1/2}} \right) \\ &\ll \sum_{\substack{I_1, \dots, I_5 \\ \mathfrak{N}I_j \leq B}} \left( \frac{2^{\omega_K(I_2) + \omega_K(I_1 I_2 I_3 I_5)} B}{\mathfrak{N}I_1^{5/4} \mathfrak{N}I_2^{3/2} \mathfrak{N}I_3^{5/4} \mathfrak{N}I_4 \mathfrak{N}I_5} + \frac{2^{\omega_K(I_2) + \omega_K(I_1 I_2 I_3 I_5)} B \log B}{\mathfrak{N}I_1^{3/2} \mathfrak{N}I_2^2 \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4 \mathfrak{N}I_5} \right) \\ &\ll B(\log B)^3 + B(\log B)^4 \ll B(\log B)^4. \end{aligned}$$

□

**Lemma 3.4.** *If  $\mathbf{I}''$  runs over all six-tuples  $(I_1, \dots, I_5, I_7)$  of nonzero ideals of  $\mathcal{O}_K$  then we have*

$$N_{86}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta'_8(\mathbf{I}'), I_6) V_{86}(\mathfrak{N}I_1, \dots, \mathfrak{N}I_5, \mathfrak{N}I_7; B) + O(B(\log B)^4).$$

*Proof.* This is analogous to [DF13, Lemma 9.4].  $\square$

### 3.2.3. The remaining summations for $N_{86}(B)$ .

**Lemma 3.5.** *We have*

$$N_{86}(B) = \pi^6 \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^8 \left( \frac{h_K}{\omega_K} \right)^6 \theta_0 V_{860}(B) + O(B(\log B)^4 \log \log B),$$

where  $\theta_0$  is as in (1.7) and

$$V_{860}(B) := \int_{t_1, \dots, t_5, t_7 \geq 1} V_{86}(t_1, \dots, t_5, t_7; B) dt_1 \cdots dt_5 dt_7,$$

with real variables  $t_1, \dots, t_5, t_7$ .

*Proof.* By [DF13, Lemma 3.5, (5)], applied to (3.5), we have, for  $t_7 \geq 1$ ,

$$V_{86}(t_1, \dots, t_5, t_7; B) \ll \frac{B}{t_1 \cdots t_5 t_7} \left( \frac{B}{t_1^3 t_2^6 t_3^4 t_4^2 t_5^5} \right)^{-1/6}.$$

Furthermore, using (3.1) to bound  $t_6$  and (3.3) to bound  $\|\eta_8\|_\infty$ , we see that

$$V_{86}(t_1, \dots, t_5, t_7; B) \ll \frac{1}{t_5} \left( \frac{B}{t_1^2 t_2^4 t_3^3 t_4^2 t_5^3} \right) \left( \frac{B}{t_1^2 t_2^3 t_3^2 t_4 t_5^2} \right) = \frac{B}{t_1 \cdots t_5 t_7} \left( \frac{B}{t_1^3 t_2^6 t_3^4 t_4^2 t_5^5} \right).$$

We apply [DF13, Proposition 7.3] with  $r = 5$ .  $\square$

### 3.2.4. The second summation over $\eta_7$ in $M_{\mathbf{C}}^{(87)}(B)$ .

**Lemma 3.6.** *Write  $\eta'' := (\eta_1, \dots, \eta_6)$ . We have*

$$M_{\mathbf{C}}^{(87)}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\eta'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}} \mathcal{A}(\theta'_8(\mathbf{I}'), I_7) V_{87}(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) + O_{\mathbf{C}}(B(\log B)^4),$$

where, for  $t_1, \dots, t_6 \geq 1$ ,

$$V_{87}(t_1, \dots, t_6; B) := \frac{\pi}{t_5} \int_{\substack{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B) \\ t_7 \geq t_6}} dt_7 d\eta_8,$$

with a real variable  $t_7$  and a complex variable  $\eta_8$ .

*Proof.* Again, we apply the strategy described in [DF13, Section 6] in the case  $b_0 \geq 2$ . However, this time we must examine the arithmetic function more carefully, since a straightforward application as in Lemma 3.3 would not yield sufficiently good error terms. We write

$$M_{\mathbf{C}}^{(87)}(B) = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\eta'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}} \sum_{\substack{\mathfrak{k}_c | I_2 \\ \mathfrak{k}_c + I_1 I_3 = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_c)}{\mathfrak{N}\mathfrak{k}_c} \Sigma, \quad (3.12)$$

where

$$\Sigma := \sum_{\substack{\eta_7 \in \mathcal{O}_{7*} \\ \mathfrak{N}I_7 \geq \mathfrak{N}I_6}} \vartheta(I_7) \sum_{\substack{\rho \mod \mathfrak{k}_c I_5 \\ \rho \mathcal{O}_K + \mathfrak{k}_c I_5 = \mathcal{O}_K \\ \rho^2 \equiv \mathfrak{k}_c I_5 \eta_6 \eta_7 A}} g(\mathfrak{N}I_7), \quad (3.13)$$

with  $\vartheta(I_7) := \tilde{\theta}_8(\mathbf{I}', \mathbf{k}_c)$  and  $g(t) := V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6, t; B)$ .

The key observation is that, as in [BD09], we can replace  $\vartheta(I_7)$  by the function

$$\vartheta'(I_7) := \theta'_0(\mathbf{I}') \frac{\phi_K^*(I_2 I_3 I_4 I_6)}{\phi_K^*(I_2 + \mathbf{k}_c I_5)},$$

where  $\theta'_0$  encodes all coprimality conditions that are encoded by  $\theta_0$ , except for allowing  $I_5 + I_7 \neq \mathcal{O}_K$ . For the representation  $\theta'_0 = \prod_{\mathfrak{p}} \theta'_{0,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}'))$  as a product of local factors, this amounts to

$$\theta'_{0,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } \theta_{0,\mathfrak{p}}(J) = 1 \text{ or } J = \{5, 7\}, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing  $\vartheta$  by  $\vartheta'$  in (3.13) does not change  $\Sigma$  for any  $\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}$  and  $\mathbf{k}_c$  as in (3.12), since the sum over  $\rho$  is zero whenever  $I_5 + I_7 \neq \mathcal{O}_K$ . Indeed, we know from Lemma 3.2 that  $\eta_6 \eta_7 A$  is invertible modulo  $\mathbf{k}_c I_5$  whenever  $\mathbf{k}_c$  is as in (3.12) and  $\theta_0(\mathbf{I}') \neq 0$ . This implies that  $v_{\mathfrak{p}}(\eta_6 A \mathcal{O}_7) = 0$  for any fixed  $\boldsymbol{\eta}'', \mathbf{k}_c$  as in (3.12) with  $\Sigma \neq 0$  and any  $\mathfrak{p} \mid \mathbf{k}_c I_5$ . Therefore, if  $p \mid I_5 + I_7$  then the second and third condition under the sum over  $\rho$  in (3.13) contradict each other.

Since  $\vartheta'(I_7) = \vartheta(I_7)$  whenever  $I_5 + I_7 = \mathcal{O}_K$ , we have  $\mathcal{A}(\vartheta'(\mathbf{a}), \mathbf{a}, \mathbf{k}_c I_5) = \mathcal{A}(\vartheta(\mathbf{a}), \mathbf{a}, \mathbf{k}_c I_5)$ .

Moreover, we obtain immediately from the definition that  $\vartheta' \in \Theta(I_1 I_2 I_3 I_4, 1, 1, 1)$  (see [DF13, Definition 2.1]). Hence, by [DF13, Lemma 2.2], the function  $\vartheta'$  satisfies [DF13, (6.1)] with  $c_{\theta} := 2^{\omega_K(I_1 I_2 I_3 I_4)} C := 0$ .

By (3.4),  $g(t) = 0$  whenever  $t > t_2 := B/(\mathfrak{N}I_1 \mathfrak{N}I_2^2 \mathfrak{N}I_3^2 \mathfrak{N}I_4^2 \mathfrak{N}I_5 \mathfrak{N}I_6^2)$ , and, by [DF13, Lemma 3.5, (2)] applied to (3.5),  $g(t) \ll B^{1/2}/(\mathfrak{N}I_1^{1/2} \mathfrak{N}I_5^{1/2}) \cdot t^{-1/2}$ . With [DF13, Proposition 6.1], we obtain

$$\begin{aligned} \Sigma &= \frac{2\pi}{\sqrt{|\Delta_K|}} \phi_K^*(\mathbf{k}_c I_5) \mathcal{A}(\vartheta(\mathbf{a}), \mathbf{a}, \mathbf{k}_c I_5) \int_{t \geq \mathfrak{N}I_6} g(t) dt \\ &\quad + O\left(\frac{2^{\omega_K(I_1 I_2 I_3 I_4)} B^{1/2}}{\mathfrak{N}I_1^{1/2} \mathfrak{N}I_5^{1/2}} \left(\sqrt{\mathfrak{N}(\mathbf{k}_c I_5)} \log B + \frac{\mathfrak{N} \mathbf{k}_c I_5}{\mathfrak{N} I_6^{1/2}} \log(\mathfrak{N}I_6 + 2)\right)\right). \end{aligned}$$

As in Lemma 3.3, the main term in the lemma is correct, and for the error term we may sum over the ideals  $\mathbf{k}_c$  and  $I_j$  instead of the  $\eta_j$ . By (3.1), (3.4), and our condition  $\mathfrak{N}I_7 \geq \mathfrak{N}I_6$ , it suffices to sum over  $\mathbf{k}_c$  and the  $(I_1, \dots, I_6)$  satisfying (3.1) and

$$\mathfrak{N}I_1^3 \mathfrak{N}I_2^6 \mathfrak{N}I_3^5 \mathfrak{N}I_4^4 \mathfrak{N}I_5^4 \mathfrak{N}I_6^4 \leq B^2. \quad (3.14)$$

Thus, the total error is bounded by

$$\begin{aligned} &\sum_{I_1, \dots, I_6} \left( \frac{2^{\omega_K(I_2) + \omega_K(I_1 I_2 I_3 I_4)} B^{1/2} \log B}{\mathfrak{N}I_1^{1/2}} + \frac{2^{\omega_K(I_2) + \omega_K(I_1 I_2 I_3 I_4)} \mathfrak{N}I_5^{1/2} B^{1/2} \log B}{\mathfrak{N}I_1^{1/2} \mathfrak{N}I_6^{1/2}} \right) \\ &\ll \sum_{\substack{I_1, \dots, I_5 \\ \mathfrak{N}I_j \leq B}} \frac{2^{\omega_K(I_2) + \omega_K(I_1 I_2 I_3 I_4)} B \log B}{\mathfrak{N}I_1^{5/4} \mathfrak{N}I_2^{3/2} \mathfrak{N}I_3^{5/4} \mathfrak{N}I_4 \mathfrak{N}I_5} + \sum_{\substack{I_1, \dots, I_4, I_6 \\ \mathfrak{N}I_j \leq B}} \frac{2^{\omega_K(I_2) + \omega_K(I_1 I_2 I_3 I_4)} B \log B}{\mathfrak{N}I_1^{3/2} \mathfrak{N}I_2^2 \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4 \mathfrak{N}I_6} \\ &\ll B(\log B)^4 + B(\log B)^4 \ll B(\log B)^4. \end{aligned}$$

□

**Lemma 3.7.** *If  $\mathbf{I}''$  runs over all six-tuples  $(I_1, \dots, I_6)$  of nonzero ideals of  $\mathcal{O}_K$  then we have*

$$N_{87}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta'_0(\mathbf{I}'), I_7) V_{87}(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) + O(B(\log B)^4).$$

*Proof.* This is analogous to [DF13, Lemma 9.4].

□

3.2.5. *The remaining summations for  $N_{87}(B)$ .*

**Lemma 3.8.** *We have*

$$N_{87}(B) = \pi^6 \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^8 \left( \frac{h_K}{\omega_K} \right)^6 \theta_0 V_{870}(B) + O(B(\log B)^4 \log \log B),$$

where  $\theta_0$  is given in (1.7) and

$$V_{870}(B) := \int_{t_1, \dots, t_6 \geq 1} V_{87}(t_1, \dots, t_6; B) dt_1 \cdots dt_6,$$

with real variables  $t_1, \dots, t_6$ .

*Proof.* By [DF13, Lemma 3.5, (6)], applied to (3.5), we have

$$V_{87}(t_1, \dots, t_6; B) \ll \frac{1}{t_5} \cdot \frac{B^{3/4} t_5^{3/4}}{t_1^{1/2} t_3^{1/4} t_4^{1/2} t_6^{3/4}} = \frac{B}{t_1 \cdots t_6} \left( \frac{B}{t_1^2 t_2^4 t_3^3 t_4^2 t_5^3 t_6} \right)^{-1/4}.$$

Furthermore, using (3.3) and (3.4) to bound  $\|\eta_8\|_\infty$  and  $t_7$ , respectively, we see that

$$V_{87}(t_1, \dots, t_6; B) \ll \frac{1}{t_5} \left( \frac{B}{t_1^2 t_2^3 t_3^2 t_5^2} \right) \left( \frac{B}{t_1 t_2^2 t_3^2 t_4^2 t_5 t_6^2} \right) = \frac{B}{t_1 \cdots t_6} \cdot \left( \frac{B}{t_1^2 t_2^4 t_3^3 t_4^2 t_5^3 t_6} \right).$$

We apply [DF13, Proposition 7.3] with  $r = 5$ .  $\square$

3.2.6. *Combining the summations.*

**Lemma 3.9.** *We have*

$$N_{U_2, H}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^8 \left( \frac{h_K}{\omega_K} \right)^6 \theta_0 V_0(B) + O(B(\log B)^4 \log \log B),$$

where  $\theta_0$  is given in (1.7) and

$$V_0(B) := \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geq 1}} \frac{1}{\|\eta_5\|_\infty} d\eta_1 \cdots d\eta_8,$$

with complex variables  $\eta_1, \dots, \eta_8$ .

*Proof.* This follows from (3.10), Lemma 3.5, and Lemma 3.8, using polar coordinates, similarly to [DF13, Lemma 9.9].  $\square$

3.3. **Proof of Theorem 1.1 for  $S_2$ .** We use the conditions

$$\|\eta_1^2 \eta_2^4 \eta_4^2 \eta_5^3 \eta_6\|_\infty \leq B \text{ and} \quad (3.15)$$

$$\|\eta_1^2 \eta_2^4 \eta_4^2 \eta_5^3 \eta_6\|_\infty \leq B \text{ and } \|\eta_1^{-1} \eta_2^{-2} \eta_4^2 \eta_5^{-3} \eta_6^4\|_\infty \leq B. \quad (3.16)$$

**Lemma 3.10.** *Let  $\alpha(\tilde{S}_2)$ ,  $\omega_\infty(\tilde{S}_2)$  be as in Theorem 1.1 and  $\mathcal{R}(B)$  as in (3.1)–(3.5). Define*

$$V'_0(B) := \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1}} \frac{1}{\|\eta_5\|_\infty} d\eta_1 \cdots d\eta_8, \quad (3.16)$$

where  $\eta_1, \dots, \eta_8$  are complex variables. Then

$$\pi^6 \alpha(\tilde{S}_2) \omega_\infty(\tilde{S}_2) B(\log B)^5 = 4V'_0(B). \quad (3.17)$$

*Proof.* The proof is analogous to the proof of Lemma 2.6. Let  $\eta_1, \eta_2, \eta_4, \eta_5, \eta_6 \in \mathbb{C}$ ,  $B > 0$ , and let  $l := (B \|\eta_1 \eta_2^2 \eta_4 \eta_5^3 \eta_6^2\|_\infty)^{1/2}$ . Let  $\eta_3, \eta_7, \eta_8$  be complex variables. Applying the coordinate transformation  $z_0 = l^{-1/3} \eta_1 \eta_2^2 \eta_4 \eta_5^2 \eta_6 \cdot \eta_3$ ,  $z_2 = l^{-1/3} \eta_1 \eta_2 \eta_5 \cdot \eta_8$ ,  $z_3 = l^{-1/3} \eta_4 \eta_6^2 \cdot \eta_7$  to  $\omega_\infty(\tilde{S}_2)$ , we obtain

$$\omega_\infty(\tilde{S}_2) = \frac{12}{\pi} \frac{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}{B} \int_{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B)} \frac{1}{\|\eta_5\|_\infty} d\eta_3 d\eta_7 d\eta_8. \quad (3.18)$$

The negative curves  $[E_1], \dots, [E_7]$  generate the effective cone of  $\tilde{S}_1$ . Because of  $[-K_{\tilde{S}_1}] = [2E_1 + 4E_2 + 3E_3 + 2E_4 + 3E_5 + E_6]$  and  $[E_7] = [E_1 + 2E_2 + E_3 + 2E_5 - E_6]$ , [DF13, Lemma 8.1] implies

$$\alpha(\tilde{S}_2)(\log B)^5 = \frac{1}{3\pi^5} \int_{\substack{\|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1 \\ (3.16)}} \frac{d\eta_1 d\eta_2 d\eta_4 d\eta_5 d\eta_6}{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}. \quad (3.19)$$

The lemma follows by substituting (3.18) and (3.19) in (3.17).  $\square$

To finish our proof, we compare  $V_0(B)$  from Lemma 3.9 with  $V'_0(B)$  defined in Lemma 3.10. Let

$$\begin{aligned} \mathcal{D}_0(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geq 1\}, \\ \mathcal{D}_1(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geq 1, (3.15)\}, \\ \mathcal{D}_2(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geq 1, (3.16)\}, \\ \mathcal{D}_3(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geq 1, (3.16)\}, \\ \mathcal{D}_4(B) &:= \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1, (3.16)\}. \end{aligned}$$

Moreover, let

$$V_i(B) := \int_{\mathcal{D}_i(B)} \frac{d\eta_1 \cdots d\eta_8}{\|\eta_5\|_\infty}.$$

Then clearly  $V_0(B)$  is as in Lemma 3.9 and  $V_4(B) = V'_0(B)$ . We show that, for  $1 \leq i \leq 4$ ,  $V_i(B) - V_{i-1}(B) = O(B(\log B)^4)$ . This holds for  $i = 1$ , since  $R_1 = R_0$ . Moreover, using [DF13, Lemma 3.5, (4)] and (3.5) to bound the integral over  $\eta_7$  and  $\eta_8$ , we have

$$V_2(B) - V_1(B) \ll \int_{\substack{\|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geq 1 \\ \|\eta_1 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5^3 \eta_6^2\|_\infty \leq B \\ \|\eta_1^{-1} \eta_2^{-2} \eta_4^2 \eta_5^{-3} \eta_6^4\|_\infty > B}} \frac{B^{3/4}}{\|\eta_1^2 \eta_3 \eta_4^2 \eta_5^3 \eta_6^3\|_\infty^{1/4}} d\eta_1 \cdots d\eta_6 \ll B(\log B)^4.$$

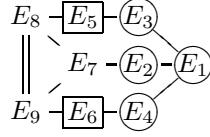
Using [DF13, Lemma 3.5, (2)] and the (3.5) to bound the integral over  $\eta_8$ , we obtain

$$V_3(B) - V_2(B) \ll \int_{\substack{\|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geq 1 \\ \|\eta_7\|_\infty < 1, (3.1), (3.16)}} \frac{B^{1/2}}{\|\eta_1 \eta_5 \eta_7\|_\infty^{1/2}} d\eta_1 \cdots d\eta_7 \ll B(\log B)^4.$$

Finally, using using [DF13, Lemma 3.5, (4)] and (3.5) to bound the integral over  $\eta_7$  and  $\eta_8$ , we have

$$V_4(B) - V_3(B) \ll \int_{\substack{\|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1 \\ \|\eta_3\|_\infty < 1, (3.15)}} \frac{B^{3/4}}{\|\eta_1^2 \eta_3 \eta_4^2 \eta_5^3 \eta_6^3\|_\infty^{1/4}} d\eta_1 \cdots d\eta_6 \ll B(\log B)^4.$$

Using Lemma 3.9 and Lemma 3.10, this implies Theorem 1.1 for  $S_2$ .

FIGURE 3. Configuration of curves on  $\tilde{S}_3$ 4. THE QUARTIC DEL PEZZO SURFACE OF TYPE  $\mathbf{D}_4$ 

4.1. **Passage to a universal torsor.** We use the notation from [Der06].

For any given  $\mathbf{C} = (C_0, \dots, C_5) \in \mathcal{C}^6$ , we define  $u_{\mathbf{C}} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_5^{-1})$  and

$$\begin{aligned} \mathcal{O}_1 &:= C_2 C_3^{-1} & \mathcal{O}_2 &:= C_1 C_2^{-1} & \mathcal{O}_3 &:= C_0 C_1^{-1} C_2^{-1} C_5^{-1} \\ \mathcal{O}_4 &:= C_3 C_4^{-1} & \mathcal{O}_5 &:= C_5 & \mathcal{O}_6 &:= C_4 \\ \mathcal{O}_7 &:= C_0 C_1^{-1} & \mathcal{O}_8 &:= C_0 C_5^{-1} & \mathcal{O}_9 &:= C_0^2 C_1^{-1} C_2^{-1} C_3^{-1} C_4^{-1} \end{aligned}$$

Let

$$\mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0}, & j \in \{1, \dots, 6\}, \\ \mathcal{O}_j, & j \in \{7, 8, 9\}. \end{cases}$$

For  $\eta_j \in \mathcal{O}_j$ , let

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

For  $B \geq 0$ , let  $\mathcal{R}(B)$  be the set of all  $(\eta_1, \dots, \eta_8) \in \mathbb{C}^8$  with  $\eta_4 \eta_6 \neq 0$  and

$$\|\eta_1^2 \eta_2 \eta_3^2 \eta_4 \eta_5^2 \eta_8\|_{\infty} \leq B, \quad (4.1)$$

$$\|\eta_1^4 \eta_2^2 \eta_3^3 \eta_4^3 \eta_5^2 \eta_6^2\|_{\infty} \leq B, \quad (4.2)$$

$$\|\eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5 \eta_6 \eta_7\|_{\infty} \leq B, \quad (4.3)$$

$$\|\eta_1^2 \eta_2 \eta_3^2 \eta_4 \eta_5^2 \eta_8 + \eta_1^2 \eta_2^2 \eta_3 \eta_4 \eta_7^2\|_{\infty} \leq B, \quad (4.4)$$

$$\left\| \frac{\eta_3 \eta_5^2 \eta_8^2 + \eta_2 \eta_7^2 \eta_8}{\eta_4 \eta_6^2} \right\|_{\infty} \leq B, \quad (4.5)$$

and let  $M_{\mathbf{C}}(B)$  be the set of all

$$(\eta_1, \dots, \eta_9) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}$$

that satisfy the *height conditions*

$$(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B),$$

the *torsor equation*

$$\eta_2 \eta_7^2 + \eta_3 \eta_5^2 \eta_8 + \eta_4 \eta_6^2 \eta_9 = 0, \quad (4.6)$$

and the *coprimality conditions*

$$I_j + I_k = \mathcal{O}_K \text{ for all distinct nonadjacent vertices } E_j, E_k \text{ in Figure 3.} \quad (4.7)$$

**Lemma 4.1.** *We have*

$$N_{U_3, H}(B) = \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} |M_{\mathbf{C}}(B)|.$$

*Proof.* Again, the lemma is a specialization of [DF13, Claim 6.1], and we prove it in an analogous way as Lemma 3.1. Starting with the curves Let  $E_3^{(0)} := \{y_1 = 0\}$ ,  $E_7^{(0)} := \{y_2 = 0\}$ ,  $E_8^{(0)} := \{y_0 = 0\}$ ,  $E_9^{(0)} := \{-y_0 y_1 - y_2^2 = 0\}$  in  $\mathbb{P}_K^2$ , we prove [DF13, Claim 6.2] for the following sequence of blow-ups:

- (1) blow up  $E_3^{(0)} \cap E_7^{(0)} \cap E_9^{(0)}$ , giving  $E_2^{(1)}$ ,
- (2) blow up  $E_2^{(1)} \cap E_3^{(1)} \cap E_9^{(1)}$ , giving  $E_1^{(2)}$ ,

- (3) blow up  $E_1^{(2)} \cap E_9^{(2)}$ , giving  $E_4^{(3)}$ ,
- (4) blow up  $E_4^{(3)} \cap E_9^{(3)}$ , giving  $E_6^{(4)}$ ,
- (5) blow up  $E_3^{(4)} \cap E_8^{(4)}$ , giving  $E_5^{(5)}$ .

The inverse  $\pi \circ \rho^{-1} : \mathbb{P}_K^2 \dashrightarrow S_3$  of the projection  $\rho \circ \pi^{-1} : S_3 \dashrightarrow \mathbb{P}_K^2$ ,  $(x_0 : \dots : x_4) \mapsto (x_0 : x_1 : x_2)$  is given by

$$(y_0 : y_1 : y_2) \mapsto (y_0 y_1^2 : y_1^3 : y_1^2 y_2 : -y_1(y_0 y_1 + y_2^2) : -y_0(y_0 y_1 + y_2^2)).$$

With the map  $\Psi$  from [DF13, Claim 4.2] sending  $(\eta_1, \dots, \eta_9)$  to

$$(\eta_1^2 \eta_2 \eta_3 \eta_4 \eta_5^2 \eta_8, \eta_1^4 \eta_2^2 \eta_3^3 \eta_4^2 \eta_5^2 \eta_6^2, \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5 \eta_6 \eta_7, \eta_1^2 \eta_2 \eta_3 \eta_4^2 \eta_6^2 \eta_9, \eta_8 \eta_9),$$

we see that the assumptions of [DF13, Lemma 4.3] are satisfied, so [DF13, Claim 4.2] holds for  $i = 0$ .

In the first two steps of the above chain of blow-ups, we are in the situation of [DF13, Remark 4.5], so certain coprimality conditions need to be checked by hand. However, up to changing some indices, our situation in steps (1) and (2) is exactly the same as in Lemma 3.1, so the arguments given there apply to our lemma as well. Steps (3), (4), (5) are again covered by [DF13, Lemma 4.4], which proves [DF13, Claim 6.2]. From this, we deduce [DF13, Claim 6.1] as in [DF13, Lemma 9.1].  $\square$

## 4.2. Summations.

4.2.1. *The first summation over  $\eta_8$  with dependent  $\eta_9$ .*

**Lemma 4.2.** *Let  $\boldsymbol{\eta}' := (\eta_1, \dots, \eta_7)$  and  $\mathbf{I}' := (I_1, \dots, I_7)$ . Then*

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{7*}} \theta_8(\mathbf{I}') V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_7; B) + O_{\mathbf{C}}(B(\log B)^2),$$

where

$$V_8(t_1, \dots, t_7; B) := \frac{1}{t_4 t_6^2} \int_{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B)} d\eta_8$$

and

$$\theta_8(\mathbf{I}') := \prod_{\mathfrak{p}} \theta_{1,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}')).$$

Here,  $J_{\mathfrak{p}}(\mathbf{I}') := \{j \in \{1, \dots, 7\} : \mathfrak{p} \mid I_j\}$  and

$$\theta_{1,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{5\}, \{6\}, \{7\}, \\ 1 - \frac{1}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 7\}, \{3, 5\}, \{4, 6\}, \\ 1 - \frac{2}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We apply [DF13, Proposition 5.3] with  $(A_1, A_0) := (2, 7)$ ,  $(B_1, B_2, B_0) := (3, 5, 8)$ ,  $(C_1, C_2, C_0) := (4, 6, 9)$ ,  $D := 1$ ,  $u_{\mathbf{C}} B$  instead of  $B$ , and  $\Pi_1, \Pi_2$  as suggested in [DF13, Remark 5.2].

As in Lemma 2.2, we see that the main arising from [DF13, Proposition 5.3] is the main term in the lemma, so it remains to deal with the error term.

For given  $\boldsymbol{\eta}'$  and  $B$ , the set of all  $\eta_8 \in \mathbb{C}$  with  $(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B)$  is contained in the union of two balls of radius

$$R(\boldsymbol{\eta}'; u_{\mathbf{C}} B) \ll_{\mathbf{C}} \begin{cases} (B \mathfrak{N}(I_4 I_6^2 I_2^{-1} I_7^{-2}))^{1/2} & \text{if } \eta_7 \neq 0, \\ (B / \mathfrak{N}(I_1^2 I_2 I_3^2 I_4 I_5^2))^{1/2} & \text{if } \eta_7 = 0. \end{cases}$$

Indeed, this follows from [DF13, Lemma 3.4, (1)], applied to (4.5), if  $\eta_7 \neq 0$  and from (4.1) if  $\eta_7 = 0$ .

Hence, the error term is

$$\ll_{\boldsymbol{\eta}'} \sum_{\boldsymbol{\eta}', (4.8), (4.9)} 2^{\omega_K(I_1 I_4) + \omega_K(I_1 I_2 I_3)} \left( \frac{R(\boldsymbol{\eta}'; u_{\mathbf{C}} B)}{\mathfrak{N}(I_4 I_6^2)^{1/2}} + 1 \right),$$

where, using (4.2), (4.3), the sum runs over all  $\boldsymbol{\eta}'$  with

$$\mathfrak{N}(I_1^4 I_2^2 I_3^3 I_4^3 I_5^2 I_6^2) \leq B, \text{ and} \quad (4.8)$$

$$\mathfrak{N}(I_1^3 I_2^2 I_3^2 I_4^2 I_5 I_6 I_7) \leq B. \quad (4.9)$$

Let us first estimate the sum over all  $\boldsymbol{\eta}'$  with  $\eta_7 \neq 0$ . We may sum over the  $I_j$  instead of the  $\eta_j$  and obtain

$$\begin{aligned} &\ll_{\mathbf{C}} \sum_{\mathbf{I}', (4.8), (4.9)} 2^{\omega_K(I_1 I_4) + \omega_K(I_1 I_2 I_3)} \left( \frac{B^{1/2}}{\mathfrak{N}(I_2 I_7^2)^{1/2}} + 1 \right) \\ &\ll \sum_{\substack{I_1, \dots, I_5, I_7 \\ \mathfrak{N}I_j \leq B}} \left( \frac{2^{\omega_K(I_1 I_4) + \omega_K(I_1 I_2 I_3)} B}{\mathfrak{N}I_1^2 \mathfrak{N}I_2^{3/2} \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4^{3/2} \mathfrak{N}I_5 \mathfrak{N}I_7} + \frac{2^{\omega_K(I_1 I_4) + \omega_K(I_1 I_2 I_3)} B}{\mathfrak{N}I_1^3 \mathfrak{N}I_2^2 \mathfrak{N}I_3^2 \mathfrak{N}I_4^2 \mathfrak{N}I_5 \mathfrak{N}I_7} \right) \\ &\ll B(\log B)^2. \end{aligned}$$

Now we assume that  $\eta_7 = 0$  and sum over the remaining variables. We obtain the upper bound

$$\begin{aligned} &\ll_{\mathbf{C}} \sum_{I_1, \dots, I_6, (4.8)} 2^{\omega_K(I_1 I_4) + \omega_K(I_1 I_2 I_3)} \left( \frac{B^{1/2}}{\mathfrak{N}I_1 \mathfrak{N}I_2^{1/2} \mathfrak{N}I_3 \mathfrak{N}I_4 \mathfrak{N}I_5 \mathfrak{N}I_6} + 1 \right) \\ &\ll \sum_{I_1, I_3, \dots, I_6} \left( \frac{2^{\omega_K(I_1 I_4) + \omega_K(I_1 I_3)} B^{3/4} \log B}{\mathfrak{N}I_1^2 \mathfrak{N}I_3^{7/4} \mathfrak{N}I_4^{7/4} \mathfrak{N}I_5^{3/2} \mathfrak{N}I_6^{3/2}} + \frac{2^{\omega_K(I_1 I_4) + \omega_K(I_1 I_3)} B^{1/2} \log B}{\mathfrak{N}I_1^2 \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4^{3/2} \mathfrak{N}I_5 \mathfrak{N}I_6} \right) \\ &\ll B^{3/4} \log B + B^{1/2} (\log B)^3 \ll B^{3/4} \log B. \end{aligned} \quad \square$$

#### 4.2.2. The second summation over $\eta_7$ .

**Lemma 4.3.** *Write  $\boldsymbol{\eta}'' := (\eta_1, \dots, \eta_6)$ . We have*

$$\begin{aligned} |M_{\mathbf{C}}(B)| &= \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}} \mathcal{A}(\theta_8(\mathbf{I}'), I_7) V_7(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) \\ &\quad + O_{\mathbf{C}}(B(\log B)^2), \end{aligned}$$

where, for  $t_1, \dots, t_6 \geq 1$ ,

$$V_7(t_1, \dots, t_6; B) := \frac{\pi}{t_4 t_6^2} \int_{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B)} dt_7 d\eta_8,$$

with a positive variable  $t_7$  and a complex variable  $\eta_8$ .

*Proof.* We apply [DF13, Proposition 6.1] as suggested in [DF13, Section 6] in the case  $b_0 = 1$ . We have

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}} \sum_{\eta_7 \in \mathcal{O}_7} \vartheta(I_7) g(\mathfrak{N}I_7) + O_{\mathbf{C}}(B(\log B)^2), \quad (4.10)$$

where  $\vartheta(I_7) := \theta_8(\mathbf{I}')$  and  $g(t) := V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6, t; B)$ .

By [DF13, Lemma 5.4, Lemma 2.2], the function  $\vartheta$  satisfies [DF13, (6.1)] with  $C := 0$  and  $c_{\vartheta} := 2^{\omega_K(I_1 I_3 \dots I_6)}$ .

By (4.3), we have  $g(t) = 0$  whenever  $t > t_2 := B/(\mathfrak{N}I_1^3\mathfrak{N}I_2^2\mathfrak{N}I_3^2\mathfrak{N}I_4^2\mathfrak{N}I_5\mathfrak{N}I_6)$ , and by [DF13, Lemma 3.4, (2)] applied to (4.5), we obtain

$$g(t) \ll \frac{1}{\mathfrak{N}I_4\mathfrak{N}I_6^2} \cdot \frac{(\mathfrak{N}I_4\mathfrak{N}I_6^2B)^{1/2}}{(\mathfrak{N}I_3\mathfrak{N}I_5^2)^{1/2}} = \frac{B^{1/2}}{\mathfrak{N}I_3^{1/2}\mathfrak{N}I_4^{1/2}\mathfrak{N}I_5\mathfrak{N}I_6} =: c_g.$$

By [DF13, Proposition 6.1], the sum over  $\eta_7$  in (4.10) is just

$$\begin{aligned} & \vartheta((0))g(0) + \frac{2\pi}{\sqrt{|\Delta_K|}} \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathcal{O}_K) \int_{t \geq 1} g(t) dt \\ & + O\left(\frac{2^{\omega_K(I_1I_3\cdots I_6)}B^{1/2}}{\mathfrak{N}I_3^{1/2}\mathfrak{N}I_4^{1/2}\mathfrak{N}I_5\mathfrak{N}I_6} \cdot \frac{B^{1/2}}{\mathfrak{N}I_1^{3/2}\mathfrak{N}I_2\mathfrak{N}I_3\mathfrak{N}I_4\mathfrak{N}I_5^{1/2}\mathfrak{N}I_6^{1/2}}\right). \end{aligned}$$

Due to (4.2),  $\vartheta((0))g(0)$  and  $\int_0^1 g(t) dt$  are dominated by the error term, so the main term in the lemma is correct.

Let us consider the error term. Both the sum and the integral are zero whenever  $\eta''$  violates (4.2). We may sum over the  $(I_1, \dots, I_6)$  satisfying (4.8) instead of the  $\eta''$ , so the error term is

$$\begin{aligned} & \ll \sum_{\mathbf{I}'', \mathfrak{N}I_j \leq B} 2^{\omega_K(I_1I_3\cdots I_6)} \left( \frac{B}{\mathfrak{N}I_1^{3/2}\mathfrak{N}I_2\mathfrak{N}I_3^{3/2}\mathfrak{N}I_4^{3/2}\mathfrak{N}I_5^{3/2}\mathfrak{N}I_6^{3/2}} \right) \\ & \ll B \log B. \end{aligned} \quad \square$$

**Lemma 4.4.** *If  $\mathbf{I}''$  runs over all six-tuples  $(I_1, \dots, I_6)$  of nonzero ideals of  $\mathcal{O}_K$  then we have*

$$N_{U_3, H}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta_8(\mathbf{I}'), I_7) V_7(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) + O(B(\log B)^2).$$

*Proof.* This is analogous to [DF13, Lemma 9.4].  $\square$

#### 4.2.3. The remaining summations.

**Lemma 4.5.** *We have*

$$N_{U_3, H}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^8 \left( \frac{h_K}{\omega_K} \right)^6 \theta_0 V_0(B) + O(B(\log B)^4 \log \log B),$$

where  $\theta_0$  is as in (1.7) and

$$V_0(B) := \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geq 1}} \frac{1}{\|\eta_4\eta_6^2\|_\infty} d\eta_1 \cdots d\eta_8,$$

with complex variables  $\eta_1, \dots, \eta_8$ .

*Proof.* By [DF13, Lemma 3.4, (5)] applied to (4.5), we have

$$V_7(t_1, \dots, t_6; B) \ll \frac{B^{3/4}}{t_2^{1/2}t_3^{1/4}t_4^{1/4}t_5^{1/2}t_6^{1/2}} = \frac{B}{t_1 \cdots t_6} \left( \frac{B}{t_1^4 t_2^2 t_3^3 t_4^3 t_5^2 t_6^2} \right)^{-1/4}.$$

We apply [DF13, Proposition 7.3] with  $r = 5$  and use polar coordinates.  $\square$

### 4.3. Proof of Theorem 1.1 for $S_3$ .

**Lemma 4.6.** *Let  $\alpha(\tilde{S}_3)$ ,  $\omega_\infty(\tilde{S}_3)$  be as in Theorem 1.1 and  $\mathcal{R}(B)$  as in (4.1)–(4.5). Define*

$$V'_0(B) := \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1 \\ \|\eta_1^4 \eta_2^2 \eta_4^3 \eta_5^2 \eta_6^2\|_\infty \leq B}} \frac{1}{\|\eta_4 \eta_6^2\|_\infty} d\eta_1 \cdots d\eta_8,$$

where  $\eta_1, \dots, \eta_8$  are complex variables. Then

$$\pi^6 \alpha(\tilde{S}_3) \omega_\infty(\tilde{S}_3) B (\log B)^5 = 4V'_0(B). \quad (4.11)$$

*Proof.* Let  $\eta_1, \eta_2, \eta_4, \eta_5, \eta_6 \in \mathbb{C}$ ,  $B > 0$ , and define  $l := (B \|\eta_1^2 \eta_2 \eta_4^3 \eta_5 \eta_6^4\|_\infty)^{1/2}$ . Let  $\eta_3, \eta_7, \eta_8$  be complex variables. After the coordinate transformation  $z_0 = l^{-1/3} \eta_5 \cdot \eta_8$ ,  $z_1 = l^{-1/3} \eta_1^2 \eta_2 \eta_4^2 \eta_5 \eta_6^2 \cdot \eta_3$ ,  $z_2 = l^{-1/3} \eta_1 \eta_2 \eta_4 \eta_6 \cdot \eta_7$ , we have

$$\omega_\infty(\tilde{S}_3) = \frac{12}{\pi} \frac{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}{B} \int_{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B)} \frac{1}{\|\eta_4 \eta_6^2\|_\infty} d\eta_3 d\eta_7 d\eta_8. \quad (4.12)$$

Since the negative curves  $[E_1], \dots, [E_6]$  generate the effective cone of  $\tilde{S}_3$ , and  $[-K_{\tilde{S}_3}] = [4E_1 + 2E_2 + 3E_3 + 3E_4 + 2E_5 + 2E_6]$ , [DF13, Lemma 8.1] gives

$$\alpha(\tilde{S}_3) (\log B)^5 = \frac{1}{3\pi^5} \int_{\substack{\|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1 \\ \|\eta_1^4 \eta_2^2 \eta_4^3 \eta_5^2 \eta_6^2\|_\infty \leq B}} \frac{d\eta_1 d\eta_2 d\eta_4 d\eta_5 d\eta_6}{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}. \quad (4.13)$$

The lemma follows by substituting (4.12) and (4.13) in (4.11).  $\square$

To finish our proof, we compare  $V_0(B)$  from Lemma 4.5 with  $V'_0(B)$  defined in Lemma 4.6. We show that, starting from  $V_0(B)$ , we can add the condition  $\|\eta_1^4 \eta_2^2 \eta_4^3 \eta_5^2 \eta_6^2\|_\infty \leq B$  and remove  $\|\eta_3\|_\infty \geq 1$  with negligible error. First, we note that (4.2), together with  $\|\eta_3\|_\infty \geq 1$  implies the condition  $\|\eta_1^4 \eta_2^2 \eta_4^3 \eta_5^2 \eta_6^2\|_\infty \leq B$ , so we can add it to the domain of integration for  $V_0(B)$  without changing the result.

Using [DF13, Lemma 3.4, (3)] applied to (4.5) to bound the integral over  $\eta_7, \eta_8$ , we see that an upper bound for  $V'_0(B) - V_0(B)$  is given by

$$\ll \int_{\substack{\|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1 \\ \|\eta_3\|_\infty < 1, \|\eta_1^4 \eta_2^2 \eta_4^3 \eta_5^2 \eta_6^2\|_\infty \leq B}} \frac{B^{3/4}}{\|\eta_2^2 \eta_3 \eta_4 \eta_5^2 \eta_6^2\|_\infty^{1/4}} d\eta_1 \cdots d\eta_6 \ll B (\log B)^4.$$

Using Lemma 4.5 and Lemma 4.6, this implies Theorem 1.1 for  $S_3$ .

## 5. THE QUARTIC DEL PEZZO SURFACE OF TYPE $\mathbf{D}_5$

**5.1. Passage to a universal torsor.** We use the notation of [Der06], except that we switch  $\eta_7$  with  $\eta_8$ .

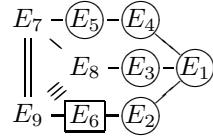


FIGURE 4. Configuration of curves on  $\tilde{S}_4$

For any given  $\mathbf{C} = (C_0, \dots, C_5) \in \mathcal{C}^6$ , we define  $u_{\mathbf{C}} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_5^{-1})$  and

$$\begin{aligned}\mathcal{O}_1 &:= C_3 C_4^{-1} & \mathcal{O}_2 &:= C_4 C_5^{-1} & \mathcal{O}_3 &:= C_0 C_1^{-1} C_2^{-1} C_3^{-1} \\ \mathcal{O}_4 &:= C_2 C_3^{-1} & \mathcal{O}_5 &:= C_1 C_2^{-1} & \mathcal{O}_6 &:= C_5 \\ \mathcal{O}_7 &:= C_0 C_1^{-1} & \mathcal{O}_8 &:= C_0 & \mathcal{O}_9 &:= C_0^3 C_1^{-1} C_2^{-1} C_3^{-1} C_4^{-1} C_5^{-1}\end{aligned}$$

Let

$$\mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0}, & i \in \{1, \dots, 6\}, \\ \mathcal{O}_j, & i \in \{7, 8, 9\}. \end{cases}$$

For  $\eta_j \in \mathcal{O}_j$ , let

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

For  $B \geq 0$ , let  $\mathcal{R}(B)$  be the set of all  $(\eta_1, \dots, \eta_8) \in \mathbb{C}^8$  with  $\eta_2 \eta_6 \neq 0$  and

$$\|\eta_1^6 \eta_2^5 \eta_3^3 \eta_4^2 \eta_5^4 \eta_6^4\|_{\infty} \leq B, \quad (5.1)$$

$$\|\eta_1^2 \eta_2 \eta_3 \eta_4^2 \eta_5^2 \eta_7^2\|_{\infty} \leq B, \quad (5.2)$$

$$\|\eta_1^4 \eta_2^3 \eta_3^2 \eta_4^3 \eta_5^2 \eta_6^2 \eta_7^2\|_{\infty} \leq B, \quad (5.3)$$

$$\|\eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5 \eta_6 \eta_8\|_{\infty} \leq B, \quad (5.4)$$

$$\left\| \frac{\eta_3 \eta_8^2 + \eta_4 \eta_5^2 \eta_7^3}{\eta_2 \eta_6^2} \right\|_{\infty} \leq B, \quad (5.5)$$

and let  $M_{\mathbf{C}}(B)$  be the set of all

$$(\eta_1, \dots, \eta_9) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}$$

that satisfy the *height conditions*

$$(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B),$$

the *torsor equation*

$$\eta_3 \eta_8^2 + \eta_2 \eta_6^2 \eta_9 + \eta_4 \eta_5^2 \eta_7^3 = 0, \quad (5.6)$$

and the *coprimality conditions*

$$I_j + I_k = \mathcal{O}_K \text{ for all distinct nonadjacent vertices } E_j, E_k \text{ in Figure 4.} \quad (5.7)$$

**Lemma 5.1.** *We have*

$$N_{U_4, H}(B) = \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} |M_{\mathbf{C}}(B)|.$$

*Proof.* The lemma is a special case of [DF13, Claim 4.1], which, as before, we prove by proving first [DF13, Claim 4.2], starting from the curves  $E_3^{(0)} := \{y_0 = 0\}$ ,  $E_7^{(0)} := \{y_1 = 0\}$ ,  $E_8^{(0)} := \{y_2 = 0\}$ ,  $E_9^{(0)} := \{-y_0 y_2^2 - y_1^3 = 0\}$  in  $\mathbb{P}_K^2$ , for the sequence of blow-ups

- (1) blow up  $E_3^{(0)} \cap E_7^{(0)} \cap E_9^{(0)}$ , giving  $E_5^{(1)}$ ,
- (2) blow up  $E_3^{(1)} \cap E_5^{(1)} \cap E_9^{(1)}$ , giving  $E_4^{(2)}$ ,
- (3) blow up  $E_3^{(2)} \cap E_4^{(2)} \cap E_9^{(2)}$ , giving  $E_1^{(3)}$ ,
- (4) blow up  $E_1^{(3)} \cap E_9^{(3)}$ , giving  $E_2^{(4)}$ ,
- (5) blow up  $E_2^{(4)} \cap E_9^{(4)}$ , giving  $E_6^{(5)}$ .

With the inverse  $\pi \circ \rho^{-1} : \mathbb{P}_K^2 \dashrightarrow S_4$  of the projection  $\rho \circ \pi^{-1} : S_4 \dashrightarrow \mathbb{P}_K^2$ ,  $(x_0 : \cdots : x_4) \mapsto (x_0 : x_2 : x_3)$  given by

$$(y_0 : y_1 : y_2) \mapsto (y_0^3 : y_0 y_1^2 : y_0^2 y_1 : y_0^2 y_2 : -(y_0 y_2^2 + y_1^3)),$$

and the map  $\Psi$  from [DF13, Claim 4.2] sending  $(\eta_1, \dots, \eta_9)$  to

$$(\eta_1^6 \eta_2^5 \eta_3^3 \eta_4^2 \eta_5^4 \eta_6^4, \eta_1^2 \eta_2 \eta_3 \eta_4^2 \eta_5^2 \eta_7^2, \eta_1^4 \eta_2^3 \eta_3^2 \eta_4^3 \eta_5^2 \eta_6^2 \eta_7, \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5 \eta_6 \eta_8, \eta_9),$$

we see that the requirements of [DF13, Lemma 4.3] are satisfied, so [DF13, Claim 6.2] holds for  $i = 0$ .

As in the proof of Lemma 3.1, we apply [DF13, Remark 4.5] for steps (1), (2), (3). For (1), we define  $\eta''_5 \in C_1$  with  $[I'_3 + I'_7 + I'_9] = [C_1^{-1}]$  such that  $I''_5 = I'_3 + I'_7 + I'_9$ . We must use the relation  $\eta''_3 \eta''_8^2 + \eta''_9 + \eta''_5 \eta''_7^3 = 0$  to check the coprimality conditions for  $\eta''_3, \eta''_5, \eta''_7, \eta''_9$ , namely  $I''_3 + I''_7 = \mathcal{O}_K$  (this holds because of the relation and  $I'_3 + I'_7 + I'_9 = \mathcal{O}_K$  by construction) and  $I''_5 + I''_7 + I''_9 = \mathcal{O}_K$  (this holds because of the relation and  $I'_3 + I'_7 + I'_9 = \mathcal{O}_K$  by construction and the coprimality condition  $I''_5 + I''_8 = \mathcal{O}_K$  provided by the proof of [DF13, Lemma 4.4]).

For (2), we define  $\eta''_4 \in C_2$  with  $[I'_3 + I'_5 + I'_9] = [C_2^{-1}]$  such that  $I''_4 = I'_3 + I'_5 + I'_9$ . The relation is  $\eta''_3 \eta''_8^2 + \eta''_9 + \eta''_4 \eta''_5 \eta''_7^3 = 0$ . We check the coprimality conditions  $I''_3 + I''_5 = \mathcal{O}_K$  (this holds because of the relation and  $I'_3 + I'_5 + I'_9 = \mathcal{O}_K$  by construction) and  $I''_5 + I''_9 = \mathcal{O}_K$  (this holds because of the relation and  $I'_3 + I'_5 = \mathcal{O}_K$  as just shown and  $I''_5 + I''_8 = \mathcal{O}_K$  by the proof of [DF13, Lemma 4.4]).

For (3), we define  $\eta''_1 \in C_3$  with  $[I'_3 + I'_4 + I'_9] = [C_3^{-1}]$  such that  $I''_1 = I'_3 + I'_4 + I'_9$ . The relation is  $\eta''_3 \eta''_8^2 + \eta''_9 + \eta''_4 \eta''_5 \eta''_7^3 = 0$ . We check the coprimality conditions are  $I''_3 + I''_4 = \mathcal{O}_K$  (this holds because of the relation and  $I'_3 + I'_4 + I'_9 = \mathcal{O}_K$  by construction),  $I''_3 + I''_9 = \mathcal{O}_K$  (this holds because of the relation and  $I'_3 + I'_4 = \mathcal{O}_K$  as just shown  $I''_3 + I''_5 = \mathcal{O}_K$  as before and  $I''_3 + I''_7 = \mathcal{O}_K$  as before) and  $I''_4 + I''_9 = \mathcal{O}_K$  (this holds because of the relation and  $I''_3 + I''_9 = \mathcal{O}_K$  as just shown and  $I''_4 + I''_8 = \mathcal{O}_K$  as before).

For (4) and (5), we can apply [DF13, Lemma 4.4]. This proves [DF13, Claim 6.2], and we deduce [DF13, Claim 6.1] as in [DF13, Lemma 9.1].  $\square$

## 5.2. Summations.

5.2.1. *The first summation over  $\eta_8$  with dependent  $\eta_9$ .* Let  $\boldsymbol{\eta}' := (\eta_1, \dots, \eta_7)$  and  $\mathbf{I}' := (I_1, \dots, I_7)$ . Let  $\theta_0(\mathbf{I}') := \prod_{\mathfrak{p}} \theta_{0,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}'))$ , where  $J_{\mathfrak{p}}(\mathbf{I}') := \{j \in \{1, \dots, 7\} : \mathfrak{p} \mid I_j\}$  and

$$\theta_{0,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \\ & \text{or } J = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 6\}, \{4, 5\}, \{5, 7\}, \\ 0 & \text{otherwise.} \end{cases}$$

We apply [DF13, Proposition 5.3] with  $(A_1, A_2, A_0) := (4, 5, 7)$ ,  $(B_1, B_0) := (3, 8)$ ,  $(C_1, C_2, C_0) := (2, 6, 9)$ , and  $D := 1$ . For given  $\eta_1, \eta_2, \eta_6$ , we write

$$\eta_4 \eta_5^2 \eta_7^3 = \eta_{A_0}^{a_0} \Pi(\boldsymbol{\eta}_A) = \Pi_1 \Pi_2^2,$$

where  $\Pi_1, \Pi_2$  are chosen as follows: Let  $\mathfrak{A} = \mathfrak{A}(\eta_1, \eta_2, \eta_6)$  be a prime ideal not dividing  $I_1 I_2 I_6$  such that  $\mathfrak{A} \mathcal{O}_7^{-1} \mathcal{O}_8 = \mathfrak{A} C_1$  is a principal fractional ideal  $t \mathcal{O}_K$ , for a suitable  $t = t(\eta_1, \eta_2, \eta_6) \in K^\times$ . Then we define  $\Pi_2 = \Pi_2(\eta_1, \eta_2, \eta_6) := \eta_7 t$  and  $\Pi_1 := \Pi_1(\eta_1, \eta_2, \eta_6) := \eta_4 \eta_5^2 \eta_7 t^{-2}$ .

**Lemma 5.2.** *We have*

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{7*}} \theta_8(\boldsymbol{\eta}', \mathbf{C}) V_8(\mathfrak{M}_1, \dots, \mathfrak{M}_7; B) + O_{\mathbf{C}}(B),$$

where

$$V_8(t_1, \dots, t_7; B) := \frac{1}{t_2 t_6^2} \int_{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B)} d\eta_8,$$

with a complex variable  $\eta_8$ . Moreover,

$$\theta_8(\boldsymbol{\eta}', \mathbf{C}) := \sum_{\substack{\mathfrak{k}_c | I_1 I_2 \\ \mathfrak{k}_c + I_3 I_4 = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_c)}{\mathfrak{N}\mathfrak{k}_c} \tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_c) \sum_{\substack{\rho \pmod{\mathfrak{k}_c I_2 I_6^2} \\ \rho \mathcal{O}_K + \mathfrak{k}_c I_2 I_6^2 = \mathcal{O}_K \\ \rho^2 \equiv_{\mathfrak{k}_c I_2 I_6^2} \eta_7 A}} 1,$$

with

$$\tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_c) := \theta_0(\mathbf{I}') \frac{\phi_K^*(I_1 I_4 I_5)}{\phi_K^*(I_1 + \mathfrak{k}_c I_2 I_6)}.$$

Here,  $A := -\eta_4 \eta_5^2 / (t(\eta_1, \eta_2, \eta_6)^2 \eta_3)$ , and  $\eta_7 A$  is invertible modulo  $\mathfrak{k}_c I_2 I_6^2$  whenever  $\theta_0(\mathbf{I}') \neq 0$ .

*Proof.* As in Lemma 3.2, we see that the main term from [DF13, Proposition 5.3] is the one given in the lemma. Let us consider the error term. For given  $\boldsymbol{\eta}'$ , the set of all  $\eta_8$  with  $(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B)$  has bounded class and is contained in two balls of radius

$$R(\boldsymbol{\eta}'; u_{\mathbf{C}} B) \ll \begin{cases} B^{3/8} \mathfrak{N}(I_2^3 I_3^{-2} I_4^{-1} I_5^{-2} I_6^6 I_7^{-3})^{1/8} & \text{if } \eta_7 \neq 0 \\ (B/\mathfrak{N}(I_1^3 I_2^2 I_3^2 I_4^2 I_5 I_6))^{1/2} & \text{if } \eta_7 = 0. \end{cases}$$

If  $\eta_7 \neq 0$  this follows from taking the geometric mean of both expressions in the minimum in [DF13, Lemma 3.5, (1)] applied to (5.5). If  $\eta_7 = 0$  then it follows from (5.4). Thus, the error term is

$$\ll \sum_{\boldsymbol{\eta}', (5.8), (5.9)} 2^{\omega_K(I_1 I_2) + \omega_K(I_1 I_4 I_5) + \omega_K(I_1 I_2 I_6)} \left( \frac{R(\boldsymbol{\eta}'; u_{\mathbf{C}} B)}{\mathfrak{N}(I_2 I_6^2)^{1/2}} + 1 \right),$$

where, using (5.1) and (5.3), the sum runs over all  $\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{7*}$  with

$$\mathfrak{N}(I_1^6 I_2^5 I_3^3 I_4^4 I_5^2 I_6^4) \leq B, \text{ and} \quad (5.8)$$

$$\mathfrak{N}(I_1^4 I_2^3 I_3^2 I_4^3 I_5^2 I_6^2 I_7) \leq B. \quad (5.9)$$

The sum over all  $\boldsymbol{\eta}'$  with  $\eta_7 \neq 0$  is bounded by

$$\begin{aligned} &\ll_{\mathbf{C}} \sum_{\mathbf{I}', (5.9)} 2^{\omega_K(I_1 I_2) + \omega_K(I_1 I_4 I_5) + \omega_K(I_1 I_2 I_6)} \left( \frac{B^{3/8}}{(\mathfrak{N} I_2 \mathfrak{N} I_3^2 \mathfrak{N} I_4 \mathfrak{N} I_5^2 \mathfrak{N} I_6^2 \mathfrak{N} I_7^3)^{1/8}} + 1 \right) \\ &\ll \sum_{\substack{I_1, \dots, I_6 \\ \mathfrak{N} I_j \leq B}} \left( \frac{2^{\omega_K(I_1 I_2) + \omega_K(I_1 I_4 I_5) + \omega_K(I_1 I_2 I_6)} B}{\mathfrak{N} I_1^{5/2} \mathfrak{N} I_2^2 \mathfrak{N} I_3^{3/2} \mathfrak{N} I_4^2 \mathfrak{N} I_5^{3/2} \mathfrak{N} I_6^{3/2}} + \frac{2^{\omega_K(I_1 I_2) + \omega_K(I_1 I_4 I_5) + \omega_K(I_1 I_2 I_6)} B}{\mathfrak{N} I_1^4 \mathfrak{N} I_2^3 \mathfrak{N} I_3^2 \mathfrak{N} I_4^3 \mathfrak{N} I_5^2 \mathfrak{N} I_6^2} \right) \\ &\ll B. \end{aligned}$$

The sum over all  $\boldsymbol{\eta}'$  with  $\eta_7 = 0$  is bounded by

$$\begin{aligned} &\ll_{\mathbf{C}} \sum_{\substack{I_1, \dots, I_6 \\ (5.8)}} 2^{\omega_K(I_1 I_2) + \omega_K(I_1 I_4 I_5) + \omega_K(I_1 I_2 I_6)} \left( \frac{B^{1/2}}{(\mathfrak{N} I_1^3 \mathfrak{N} I_2^3 \mathfrak{N} I_3^2 \mathfrak{N} I_4^2 \mathfrak{N} I_5^3 \mathfrak{N} I_6^3)^{1/2}} + 1 \right) \\ &\ll \sum_{\substack{I_1, \dots, I_6 \\ \mathfrak{N} I_j \leq 1}} 2^{\omega_K(I_1 I_2)} \left( \frac{2^{\omega_K(I_1 I_4 I_5) + \omega_K(I_1 I_2 I_6)} B^{3/4}}{\mathfrak{N} I_1^3 \mathfrak{N} I_2^{11/4} \mathfrak{N} I_3^{7/4} \mathfrak{N} I_4^2 \mathfrak{N} I_6^{5/2}} + \frac{2^{\omega_K(I_1 I_4 I_5) + \omega_K(I_1 I_2 I_6)} B^{1/2}}{\mathfrak{N} I_1^3 \mathfrak{N} I_2^{5/2} \mathfrak{N} I_3^{3/2} \mathfrak{N} I_4^2 \mathfrak{N} I_6^2} \right) \\ &\ll B^{3/4} + B^{1/2} \ll B^{3/4}. \end{aligned} \quad \square$$

5.2.2. *The second summation over  $\eta_7$ .* We define

$$\theta'_8(\mathbf{I}') := \sum_{\substack{\mathbf{k}_c | I_1 I_2 \\ \mathbf{k}_c + I_3 I_4 = \mathcal{O}_K}} \frac{\mu_K(\mathbf{k}_c)}{\mathfrak{N}\mathbf{k}_c} \tilde{\theta}_8(\mathbf{I}', \mathbf{k}_c).$$

**Lemma 5.3.** *Write  $\boldsymbol{\eta}'' := (\eta_1, \dots, \eta_6)$ . We have*

$$|M_{\mathbf{C}}(B)| = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}} \mathcal{A}(\theta'_8(\mathbf{I}'), I_7) V_7(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) + O_{\mathbf{C}}(B(\log B)^3),$$

where, for  $t_1, \dots, t_6 \geq 1$ ,

$$V_7(t_1, \dots, t_6; B) := \frac{\pi}{t_2 t_6^2} \int_{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B)} dt_7 d\eta_8,$$

with a positive variable  $t_7$  and a complex variable  $\eta_8$ .

*Proof.* We write

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \dots \times \mathcal{O}_{6*}} \sum_{\substack{\mathbf{k}_c | I_1 I_2 \\ \mathbf{k}_c + I_3 I_4 = \mathcal{O}_K}} \frac{\mu_K(\mathbf{k}_c)}{\mathfrak{N}\mathbf{k}_c} \Sigma + O_{\mathbf{C}}(B),$$

where

$$\Sigma := \sum_{\eta_7 \in \mathcal{O}_7} \vartheta(I_7) \sum_{\substack{\rho \mod \mathbf{k}_c I_2 I_6^2 \\ \rho \mathcal{O}_K + \mathbf{k}_c I_2 I_6^2 = \mathcal{O}_K \\ \rho^2 \equiv_{\mathbf{k}_c I_2 I_6^2} \eta_7 A}} g(\mathfrak{N}I_7),$$

with  $\vartheta(I_7) := \tilde{\theta}_8(\mathbf{I}', \mathbf{k}_c)$  and  $g(t) := V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6, t; B)$ . As in Lemma 3.3, the function  $\vartheta$  satisfies [DF13, (6.1)] with  $C := 0$ ,  $c_{\vartheta} := 2^{\omega_K(I_1 I_2 I_3 I_4 I_6)}$ . Moreover, by (5.2), we have  $g(t) = 0$  whenever  $t > t_2 := B^{1/2}/(\mathfrak{N}I_1 \mathfrak{N}I_2^{1/2} \mathfrak{N}I_3^{1/2} \mathfrak{N}I_4 \mathfrak{N}I_5)$ , and by [DF13, Lemma 3.5, (2)] applied to (5.5), we obtain  $g(t) \ll B^{1/2}/(\mathfrak{N}I_2^{1/2} \mathfrak{N}I_3^{1/2} \mathfrak{N}I_6)$ .

By [DF13, Proposition 6.1], we have

$$\begin{aligned} \Sigma &= \vartheta((0))g(0) + \frac{2\pi}{\sqrt{|\Delta_K|}} \phi_K^*(\mathbf{k}_c I_2 I_6^2) \mathcal{A}(\vartheta(\mathbf{a}), \mathbf{a}, \mathbf{k}_c I_2 I_6^2) \int_{t \geq 1} g(t) dt \\ &\quad + O\left(\frac{2^{\omega_K(I_1 I_2 I_3 I_4 I_6)} B^{1/2}}{\mathfrak{N}I_2^{1/2} \mathfrak{N}I_3^{1/2} \mathfrak{N}I_6} \left( \frac{B^{1/4} \mathfrak{N}\mathbf{k}_c^{1/2} \mathfrak{N}I_2^{1/4} \mathfrak{N}I_6}{\mathfrak{N}I_1^{1/2} \mathfrak{N}I_3^{1/4} \mathfrak{N}I_4^{1/2} \mathfrak{N}I_5^{1/2}} + \mathfrak{N}(\mathbf{k}_c I_2 I_6^2) \log B \right)\right), \end{aligned}$$

where the contribution of  $\vartheta((0))g(0)$  and of  $\int_0^1 g(t) dt$  is dominated by the error term. Using [DF13, Lemma 6.3], we see that this gives the correct main term in the lemma.

Summing the error term over  $I_j$  and  $\mathbf{k}_c$ , we obtain an upper bound

$$\begin{aligned} &\sum_{\substack{I_1, \dots, I_6 \\ (5.8)}} \left( \frac{2^{\omega_K(I_1 I_2) + \omega_K(I_1 \dots I_4 I_6)} B^{3/4}}{\mathfrak{N}I_1^{1/2} \mathfrak{N}I_2^{1/4} \mathfrak{N}I_3^{3/4} \mathfrak{N}I_4^{1/2} \mathfrak{N}I_5^{1/2}} + \frac{2^{\omega_K(I_1 I_2) + \omega_K(I_1 \dots I_4 I_6)} B^{1/2} \log B}{\mathfrak{N}I_2^{-1/2} \mathfrak{N}I_3^{1/2} \mathfrak{N}I_6^{-1}} \right) \\ &\ll \sum_{\substack{I_1, \dots, I_5 \\ \mathfrak{N}I_j \leq B}} \left( \frac{2^{\omega_K(I_1 I_2) + \omega_K(I_1 \dots I_4)} B \log B}{\mathfrak{N}I_1^2 \mathfrak{N}I_2^{3/2} \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4^{3/2} \mathfrak{N}I_5} + \frac{2^{\omega_K(I_1 I_2) + \omega_K(I_1 \dots I_4)} B (\log B)^2}{\mathfrak{N}I_1^3 \mathfrak{N}I_2^2 \mathfrak{N}I_3^2 \mathfrak{N}I_4^2 \mathfrak{N}I_5} \right) \\ &\ll B(\log B)^2 + B(\log B)^3 \ll B(\log B)^3. \end{aligned}$$

□

**Lemma 5.4.** *If  $\mathbf{I}''$  runs over all six-tuples  $(I_1, \dots, I_6)$  of nonzero ideals of  $\mathcal{O}_K$  then we have*

$$N_{U_4, H}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta'_8(\mathbf{I}'), I_7) V_7(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) + O(B(\log B)^4).$$

*Proof.* This is analogous to [DF13, Lemma 9.4].  $\square$

### 5.2.3. The remaining summations.

**Lemma 5.5.** *We have*

$$N_{U_4, H}(B) = \left( \frac{2}{\sqrt{|\Delta_K|}} \right)^8 \left( \frac{h_K}{\omega_K} \right)^6 \theta_0 V_0(B) + O(B(\log B)^4 \log \log B),$$

where  $\theta_0$  is as in (1.7) and

$$V_0(B) := \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geq 1}} \frac{1}{\|\eta_2 \eta_6^2\|_\infty} d\eta_1 \cdots d\eta_8,$$

with complex variables  $\eta_1, \dots, \eta_8$ .

*Proof.* By [DF13, Lemma 3.5, (5)] applied to (5.5), we have

$$V_7(t_1, \dots, t_6; B) \ll \frac{B^{5/6}}{t_2^{1/6} t_3^{1/2} t_4^{1/3} t_5^{2/3} t_6^{1/3}} = \frac{B}{t_1 \cdots t_6} \left( \frac{B}{t_1^6 t_2^5 t_3^3 t_4^4 t_5^2 t_6^4} \right)^{-1/6}.$$

We apply [DF13, Proposition 7.3] with  $r = 5$  and use polar coordinates.  $\square$

### 5.3. Proof of Theorem 1.1 for $S_4$ .

**Lemma 5.6.** *Let  $\alpha(\tilde{S}_4)$ ,  $\omega_\infty(\tilde{S}_4)$  be as in Theorem 1.1,  $\mathcal{R}(B)$  as in (5.1)–(5.5), and define*

$$V'_0(B) := \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1 \\ \|\eta_1^3 \eta_2^5 \eta_4^4 \eta_5^2 \eta_6^4\|_\infty \leq B}} \frac{1}{\|\eta_2 \eta_6^2\|_\infty} d\eta_1 \cdots d\eta_8,$$

where  $\eta_1, \dots, \eta_8$  are complex variables. Then

$$\pi^6 \alpha(\tilde{S}_4) \omega_\infty(\tilde{S}_4) B(\log B)^5 = 4V'_0(B). \quad (5.10)$$

*Proof.* Let  $\eta_1, \eta_2, \eta_4, \eta_5, \eta_6 \in \mathbb{C}$ ,  $B > 0$ , and  $l := (B \|\eta_1^3 \eta_2^5 \eta_4^4 \eta_5^2 \eta_6^4\|_\infty)^{1/2}$ . Let  $\eta_3, \eta_7, \eta_8$  be complex variables. We apply the coordinate transformation  $z_0 = l^{-1/3} \eta_1^3 \eta_2^5 \eta_4^4 \eta_5^2 \eta_6^4 \cdot \eta_3$ ,  $z_2 = l^{-1/3} \eta_1 \eta_2 \eta_4 \eta_5 \eta_6 \cdot \eta_7$ ,  $z_3 = l^{-1/3} \cdot \eta_8$  to  $\omega_\infty(\tilde{S}_4)$  and obtain

$$\omega_\infty(\tilde{S}_4) = \frac{12}{\pi} \frac{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}{B} \int_{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B)} \frac{1}{\|\eta_2 \eta_6^2\|_\infty} d\eta_3 d\eta_7 d\eta_8. \quad (5.11)$$

Since the negative curves  $[E_1], \dots, [E_6]$  generate the effective cone of  $\tilde{S}_4$ , and  $[-K_{\tilde{S}_4}] = [6E_1 + 5E_2 + 3E_3 + 4E_4 + 2E_5 + 4E_6]$ , [DF13, Lemma 8.1] gives

$$\alpha(\tilde{S}_4)(\log B)^5 = \frac{1}{3\pi^5} \int_{\substack{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \\ \|\eta_1^6 \eta_2^5 \eta_4^4 \eta_5^2 \eta_6^4\|_\infty \leq B}} \frac{d\eta_1 d\eta_2 d\eta_4 d\eta_5 d\eta_6}{\|\eta_1 \eta_2 \eta_4 \eta_5 \eta_6\|_\infty}. \quad (5.12)$$

The lemma follows by substituting (5.11) and (5.12) in (5.10).  $\square$

To finish our proof, we compare  $V_0(B)$  defined in Lemma 5.5 with  $V'_0(B)$  defined in Lemma 5.6. Starting from  $V_0(B)$ , we can add the condition  $\|\eta_1^6\eta_2^5\eta_4^4\eta_5^2\eta_6^4\|_\infty \leq B$  and remove  $\|\eta_3\|_\infty \geq 1$  with negligible error. Indeed, adding the condition  $\|\eta_1^6\eta_2^5\eta_4^4\eta_5^2\eta_6^4\|_\infty \leq B$  to the domain of integration for  $V_0(B)$  does not change the result. Using [DF13, Lemma 3.5, (3)] applied to (5.5) to bound the integral over  $\eta_7, \eta_8$ , we see that  $V'_0(B) - V_0(B)$  is

$$\ll \int_{\substack{\|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geq 1 \\ \|\eta_3\|_\infty < 1 \\ \|\eta_1^6\eta_2^5\eta_4^4\eta_5^2\eta_6^4\|_\infty \leq B}} \frac{B^{5/6}}{\|\eta_2\eta_3^3\eta_4^2\eta_5^4\eta_6^2\|_\infty^{1/6}} d\eta_1 \cdots d\eta_6 \ll B(\log B)^4.$$

Using Lemma 5.5 and Lemma 5.6, this implies Theorem 1.1 for  $S_4$ .

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